

GENERALIZED MORREY SPACES AND TRACE OPERATOR

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ABSTRACT. The theory of generalized Besov-Morrey spaces and generalized Triebel-Lizorkin-Morrey spaces is developed. Generalized Morrey spaces, which T. Mizuhara and E. Nakai proposed, are equipped with a parameter and a function. The trace property is one of the main focuses of the present paper, which will clarify the role of the parameter of generalized Morrey spaces. The quarkonial decomposition is obtained as an application of atomic decomposition. In the end, the relation between the function spaces dealt in the present paper and the foregoing researches is discussed.

1. INTRODUCTION

In the present paper, we systematically develop the theory of generalized Besov-Morrey spaces and generalized Triebel-Lizorkin-Morrey spaces and then we compare our results with existing ones in Section 7. Our results will polish existing ones, as is seen from Section 7.

Let $0 < q \leq p < \infty$. Then, the Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ is the set of all measurable functions f for which the quasi-norm

$$\|f\|_{\mathcal{M}_q^p} \equiv \sup_{Q \in \mathcal{D}} |Q|^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q |f(y)|^q dy \right)^{\frac{1}{q}}$$

is finite, where \mathcal{D} denotes the set of all dyadic cubes.

In the present paper, we consider the role of the parameter q in $\mathcal{M}_q^p(\mathbb{R}^n)$ by considering the generalized Morrey space $\mathcal{M}_q^\varphi(\mathbb{R}^n)$.

Definition 1.1. [39] Let $0 < q < \infty$ and $\varphi : (0, \infty) \rightarrow (0, \infty)$ be a function. Then define

$$\|f\|_{\mathcal{M}_q^\varphi} \equiv \sup_{Q \in \mathcal{D}} \varphi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q |f(y)|^q dy \right)^{\frac{1}{q}}$$

for a measurable function f . The space $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ is the set of all measurable functions f for which the quasi-norm $\|f\|_{\mathcal{M}_q^\varphi}$ is finite.

We envisage the following functions as examples of φ :

Example 1.2. Let $0 < q \leq p < \infty$ and $\varphi : (0, \infty) \rightarrow (0, \infty)$ be a function.

- (1) We can recover the Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ by letting $\varphi(t) = t^{n/p}$ for $t > 0$. We discuss why we need to generalize the parameter p in Proposition 7.3 and Remark 7.4.
- (2) We can recover the Lebesgue space $L^q(\mathbb{R}^n)$ by letting $\varphi(t) = t^{n/q}$ for $t > 0$.
- (3) A simple but standard example is as follows:

$$\varphi(t) = \frac{t}{l_n(t)} \in \mathcal{G}_1 \equiv \bigcap_{0 < s < t < \infty} \{ \varphi : (0, \infty) \rightarrow (0, \infty) : \varphi(s) \leq \varphi(t), \varphi(s)s^{-n} \geq \varphi(t)t^{-n} \},$$

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where $l_n(t)$ is given inductively by:

$$l_0(t) = t, \quad l_n(t) = \log(3 + l_{n-1}(t)) \quad (n = 1, 2, \dots)$$

for $t > 0$.

We shall establish that the parameter q in $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ plays the role of local regularity by considering the trace property of generalized Besov-Morrey spaces and generalized Triebel-Lizorkin-Morrey spaces. To define these spaces, we use the following notation in the present paper:

- By “cube” we mean a compact cube whose edges are parallel to the coordinate *axes*. If a cube has center x and side-length r , we denote it by $Q(x, r)$. From the definition of $Q(x, r)$,

$$(1.1) \quad |Q(x, r)| = (2r)^n.$$

We write $Q(r)$ instead of $Q(0, r)$. Conversely, given a cube Q , we denote by $c(Q)$ the center of Q and by $\ell(Q)$ the side-length of Q : $\ell(Q) = |Q|^{1/n}$, where $|Q|$ denotes the volume of the cube Q .

- Let $a \in \mathbb{R}$. Then write $a_+ \equiv \max(a, 0)$ and $a_- \equiv \min(a, 0)$. The Gauss sign $[a]$ is defined to be the largest integer m which is less than or equal to a .
- Let $A, B \geq 0$. Then $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq CB$, where C depends only on the parameters of importance. When $A \lesssim B \lesssim A$, write $A \sim B$. When we want to stress that the implicit constants in these symbols depend on important parameters, add them as subscripts. For example, $A \lesssim_p B$ means that there exists a constant $C > 0$ depending only on p such that $A \leq CB$.
- Let $a \in \mathbb{R}^n$. We define $\langle a \rangle \equiv \sqrt{1 + |a|^2}$.
- For $N \in \mathbb{N}$ and $\varphi \in C^N(\mathbb{R}^n)$, one defines

$$p_N(\varphi) \equiv \sum_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha \varphi(x)|.$$

- The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is defined to be the set of all $f \in C^\infty(\mathbb{R}^n)$ for which the semi-norm $p_N(f)$ is finite for all $N \in \mathbb{N}_0$.
- The space $\mathcal{S}_\infty(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}(\mathbb{R}^n)$ for which

$$\int_{\mathbb{R}^n} x^\alpha f(x) dx = 0$$

for all $\alpha \in \mathbb{N}_0^n$.

- The topological dual of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}_\infty(\mathbb{R}^n)$ are denoted by $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}'_\infty(\mathbb{R}^n)$, respectively. Equip $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}'_\infty(\mathbb{R}^n)$ with the weak-* topology.
- For $j \in \mathbb{Z}$ and $m = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$, we define $Q_{jm} \equiv \prod_{k=1}^n \left[\frac{m_k}{2^j}, \frac{m_k + 1}{2^j} \right)$. If notational confusion seems likely, we write $Q_{j,m}$ instead of Q_{jm} . Denote by $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ the set of such cubes. The elements in \mathcal{D} are called dyadic cubes. In the present paper, \mathcal{D} does not stand for the set of all compactly supported functions $C_c^\infty(\mathbb{R}^n)$.
- Define the Fourier transform and its inverse by

$$\begin{cases} \mathcal{F}f(\xi) \equiv \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx & (\xi \in \mathbb{R}^n) \\ \mathcal{F}^{-1}f(x) \equiv \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} f(\xi) e^{ix \cdot \xi} d\xi & (x \in \mathbb{R}^n) \end{cases}$$

if f is an integrable function. In a standard way, we extend the definition of \mathcal{F} and \mathcal{F}^{-1} to the space of all tempered distributions $\mathcal{S}'(\mathbb{R}^n)$.

- For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we write $\varphi(D)f \equiv \mathcal{F}^{-1}[\varphi \mathcal{F}f]$, or equivalently we define

$$\varphi(D)f(x) \equiv \frac{1}{\sqrt{(2\pi)^n}} \langle f, \mathcal{F}^{-1}\varphi(x - \cdot) \rangle.$$

- The Kronecker delta is given by:

$$\delta_{ab} \equiv \begin{cases} 1 & a = b, \\ 0 & a \neq b \end{cases}$$

for $a, b \in \mathbb{Z}$.

- We make use of the following notation: for $m = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$, we define $m' \equiv (m_1, m_2, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$. Conversely, we shall write $m = (m', m_n)$ for $m' = (m_1, m_2, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$ and $m_n \in \mathbb{Z}$.
- Denote by $\text{BC}(\mathbb{R}^n)$ the Banach space of all bounded continuous functions. Let $f \in \text{BC}(\mathbb{R}^n)$. Then we define $\|f\|_{\text{BC}}$ by $\|f\|_{\text{BC}} \equiv \|f\|_{L^\infty}$.
- Let $m \in \mathbb{N}_0$. Then, denote by $\text{BC}^m(\mathbb{R}^n)$ the linear space of functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ such that $f \in C^m$ and $\partial^\alpha f \in \text{BC}$ for any multi-index α with $|\alpha| \leq m$. We define the norm such that

$$\|f\|_{\text{BC}^m} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{\text{BC}}.$$

- Denote by $\text{BUC}(\mathbb{R}^n)$ the Banach space consisting of bounded uniformly continuous functions. Then we define $\|f\|_{\text{BUC}} \equiv \|f\|_\infty$.
- Let $\nu > 0$. The space $H_2^\nu(\mathbb{R}^n)$ stands for the (L^2 -based) potential space of order ν ;

$$H_2^\nu(\mathbb{R}^n) \equiv \{H \in \mathcal{S}'(\mathbb{R}^n) : (1 - \Delta)^{\nu/2} H \in L^2(\mathbb{R}^n)\}.$$

Equip $H_2^\nu(\mathbb{R}^n)$ with the norm:

$$\|H\|_{H_2^\nu} \equiv \|(1 - \Delta)^{\nu/2} H\|_2 \quad (H \in H_2^\nu(\mathbb{R}^n)).$$

- Let K be a compact set. The set $\mathcal{S}'_K(\mathbb{R}^n)$ denotes the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\mathcal{F}f$ is supported on K . Likewise define $\mathcal{S}_K(\mathbb{R}^n) \equiv \mathcal{S}(\mathbb{R}^n) \cap \mathcal{S}'_K(\mathbb{R}^n)$.
- When two Banach spaces X and Y are isomorphic, write $X \simeq Y$.

Now let us define generalized Besov-Morrey spaces and generalized Triebel-Lizorkin-Morrey spaces. Let $0 < q < \infty$. Denote by \mathcal{G}_q the set of all nondecreasing functions $\varphi : (0, \infty) \rightarrow (0, \infty)$ such that

$$(1.2) \quad \varphi(t_1)t_1^{-n/q} \geq \varphi(t_2)t_2^{-n/q} \quad (0 < t_1 \leq t_2 < \infty).$$

Definition 1.3. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$. Let θ and τ be compactly supported functions satisfying

$$0 \notin \text{supp}(\tau), \quad \theta(\xi) > 0 \text{ if } \xi \in Q(2), \quad \tau(\xi) > 0 \text{ if } \xi \in Q(2) \setminus Q(1).$$

Define $\tau_k(\xi) \equiv \tau(2^{-k}\xi)$ for $\xi \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

- (1) The (nonhomogeneous) generalized Besov-Morrey space $\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm

$$(1.3) \quad \|f\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s} \equiv \begin{cases} \left\| \theta(D)f\|_{\mathcal{M}_q^\varphi} + \left(\sum_{j=1}^{\infty} 2^{jsr} \|\tau_j(D)f\|_{\mathcal{M}_q^\varphi}^r \right)^{\frac{1}{r}} & (r < \infty), \\ \left\| \theta(D)f\|_{\mathcal{M}_q^\varphi} + \sup_{j \in \mathbb{N}} 2^{js} \|\tau_j(D)f\|_{\mathcal{M}_q^\varphi} & (r = \infty) \end{cases}$$

is finite.

- (2) The (nonhomogeneous) generalized Triebel-Lizorkin-Morrey space $\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm

$$(1.4) \quad \|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} \equiv \begin{cases} \|\theta(D)f\|_{\mathcal{M}_q^\varphi} + \left\| \left(\sum_{j=1}^{\infty} 2^{jsr} |\tau_j(D)f|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi} & (r < \infty), \\ \|\theta(D)f\|_{\mathcal{M}_q^\varphi} + \left\| \sup_{j \in \mathbb{N}} 2^{js} |\tau_j(D)f| \right\|_{\mathcal{M}_q^\varphi} & (r = \infty) \end{cases}$$

is finite.

- (3) The space $\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ denotes either $\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ or $\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$.

The next theorem answers the most fundamental question on these function spaces: do the definitions of $\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ depend on the different choices of admissible θ and τ ?

Theorem 1.4. *Let $0 < q \leq r < \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$. Assume that there exist constants $\varepsilon > 0$ and $C > 0$ such that*

$$\frac{t^\varepsilon}{\varphi(t)} \leq C \frac{r^\varepsilon}{\varphi(r)} \quad (t \geq r),$$

in the case when $\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) = \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ with $r < \infty$. Then different choices of admissible θ and τ will yield equivalent norms.

Theorem 1.4 is a starting point of the present paper. Based upon this result, we investigate the decomposition properties and the fundamental theorems.

In the present paper we investigate the role of the parameter q in $\mathcal{M}_q^\varphi(\mathbb{R}^n)$. An experience in [7, 63] shows that a passage from the classical Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ to the generalized Morrey space $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ is not a mere quest to generalization. It naturally emerges when we consider the limiting case of the Sobolev embedding; see [7, 63].

We structure the remaining part of the present paper as follows: Section 2 reviews the fundamental property of the underlying space $\mathcal{M}_q^\varphi(\mathbb{R}^n)$; we transform the results obtained earlier to a form we use in the present paper. In Section 3, we justify the definition of generalized Besov spaces and generalized Triebel-Lizorkin-Morrey spaces on \mathbb{R}^n . We prove Theorem 1.4. Theorem 1.4 can be generalized in many directions; see Sections 7.3 and 7.5 for some hints to generalize what we obtain. We also investigate some fundamental properties. Section 4 considers decompositions; atomic decomposition, molecular decomposition and quarkonial decomposition will be obtained. As applications of these results, in Section 5, we establish fundamental theorems in these function spaces. The first one is the boundedness of the trace operator, which is new. In the general setting, it is difficult to describe the image of the trace operator. Next, we investigate the pointwise multiplication property. Finally, we consider the diffeomorphism properties. The pointwise multiplication property is partially obtained in [30, Section 5]. However, our result will be sharper; in the earlier work [30] the authors depended on the Peetre maximal operator but in the present paper we do not have to rely upon this maximal operator. For the definition of the Peetre maximal operator, see Lemma 7.1 below. Comparing [60, Theorems 4.9 and 4.12] with [30, Section 4], we see that the postulates in the theorems of decomposition in [60, Theorems 4.9 and 4.12] can be milder than those in [30, Section 4]. Our results carry over to homogeneous spaces, which will be done in Section 6. We discuss a property of the topologies of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}_\infty(\mathbb{R}^n)$ as well in Section 6. Finally, in Section 7, we describe the recent development of the related function spaces and compare our results with the ones obtained earlier. In particular, let us recall that we used the following notation in [60]:

$$\mathcal{N}_{pqr}^s(\mathbb{R}^n) = \mathcal{N}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n), \quad \mathcal{E}_{pqr}^s(\mathbb{R}^n) = \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n), \quad \mathcal{A}_{pqr}^s(\mathbb{R}^n) = \mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$$

when $\varphi(t) = t^{n/p}$, so that this paper will reinforce [60]. Of course, we can recover the Besov space $B_{pq}^s(\mathbb{R}^n)$ and the Triebel-Lizorkin space $F_{pq}^s(\mathbb{R}^n)$ for any $0 < p, q \leq \infty$ and $s \in \mathbb{R}$:

$$B_{pq}^s(\mathbb{R}^n) = \mathcal{N}_{ppq}^s(\mathbb{R}^n), \quad F_{pq}^s(\mathbb{R}^n) = \mathcal{E}_{ppq}^s(\mathbb{R}^n),$$

respectively. As a consequence, the notation $A_{pq}^s(\mathbb{R}^n)$ agrees with $\mathcal{A}_{ppq}^s(\mathbb{R}^n)$; see [60]. We will discuss what results were obtained earlier and where our results in the present papers can be located. Based on the results obtained here, we will discuss some possible extensions of the results.

2. STRUCTURE OF $\mathcal{M}_q^\varphi(\mathbb{R}^n)$

2.1. Assumptions on φ . The next lemma justifies our class \mathcal{G}_q .

Lemma 2.1. *Let $0 < q < \infty$.*

- (1) [40, p.446] *For all $\varphi : (0, \infty) \rightarrow (0, \infty)$, there exists $\varphi^* \in \mathcal{G}_q$ such that*

$$\mathcal{M}_q^\varphi(\mathbb{R}^n) \simeq \mathcal{M}_q^{\varphi^*}(\mathbb{R}^n)$$

in the sense of norm equivalence.

- (2) *For any function $\varphi : (0, \infty) \rightarrow (0, \infty)$, $\mathcal{M}_q^\varphi \neq \{0\}$ if and only if*

$$\varphi^*(t) \equiv \sup_{s \in [t, \infty)} t^{\frac{n}{q}} s^{-\frac{n}{q}} \varphi(s)$$

is finite for all $t > 0$.

Proof. (1) is known, cf. [40, p.446]. The “if part” of (2) is trivial, because we know that

$$\|\chi_{[0, t]^n}\|_{\mathcal{M}_q^\varphi} = \varphi(t)$$

for $t > 0$. So, we concentrate on the “only if part” of (2).

Assume $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ contains a nonzero function; $f \in \mathcal{M}_q^\varphi \setminus \{0\}$ and suppose to the contrary $\varphi^*(t) = \infty$. Then there exists $x_f \in \mathbb{R}^n$ such that

$$\int_{Q(x_f, t)} |f(y)|^q dy > 0.$$

For each $m \in \mathbb{N}$, we can find $s_m \in [t, \infty)$ such that $t^{\frac{n}{q}} s_m^{-\frac{n}{q}} \varphi(s_m) > m$. Therefore,

$$\begin{aligned} \|f\|_{\mathcal{M}_q^\varphi} &\geq \sup_{r>0} \varphi(r) \left(\frac{1}{r^n} \int_{Q(x_f, r)} |f(y)|^q dy \right)^{1/q} \\ &\geq \varphi(s_m) s_m^{-\frac{n}{q}} \left(\int_{Q(x_f, t)} |f(y)|^q dy \right)^{\frac{1}{q}} \\ &\geq m \left(\frac{1}{t^n} \int_{Q(x_f, t)} |f(y)|^q dy \right)^{\frac{1}{q}}. \end{aligned}$$

Since m is arbitrary and independent of t , this contradicts to $f \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$. \square

The next lemma ensures that $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ contains a nonzero function.

Lemma 2.2. [6, Proposition A] *Let $0 < q < \infty$ and $\varphi \in \mathcal{G}_q$. Then*

$$(2.1) \quad \|\chi_{Q(x, R)}\|_{\mathcal{M}_q^\varphi} = \varphi(R)$$

for all $x \in \mathbb{R}^n$ and $R > 0$.

A direct consequence of the above quantitative information is:

Corollary 2.3. *Let $0 < q < \infty$ and $\varphi : (0, \infty) \rightarrow (0, \infty)$ be a function in the class \mathcal{G}_q . If $N_0 > n/q$, then $(1 + |\cdot|)^{-N_0} \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$.*

Proof. Since $\varphi \in \mathcal{G}_q$, we have $\varphi(t)t^{-n/q} \leq \varphi(1)$ for all $t \geq 1$. we have

$$\|(1 + |\cdot|)^{-N_0}\|_{\mathcal{M}_q^\varphi} \lesssim \sum_{j=1}^{\infty} \max(1, j-1)^{-N_0} \|\chi_{Q(j)}\|_{\mathcal{M}_q^\varphi} \leq \sum_{j=1}^{\infty} \max(1, j-1)^{-N_0} \varphi(j) < \infty.$$

Here for the second inequality we invoked (2.1). \square

We also use the following inequality:

Lemma 2.4. *Let $0 < q < \infty$ and $\varphi : (0, \infty) \rightarrow (0, \infty)$ be a function. Then*

$$(\|f + g\|_{\mathcal{M}_q^\varphi})^{\min(1, q)} \leq (\|f\|_{\mathcal{M}_q^\varphi})^{\min(1, q)} + (\|g\|_{\mathcal{M}_q^\varphi})^{\min(1, q)}$$

for all $f, g \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$.

We also verify the relation between f and $|f|^u$ in the next lemma.

Lemma 2.5. *Let $0 < u, q < \infty$ and $\varphi : (0, \infty) \rightarrow (0, \infty)$. Then*

$$\||f|^u\|_{\mathcal{M}_q^\varphi} = \left(\|f\|_{\mathcal{M}_{uq}^{\varphi^{1/u}}} \right)^u$$

for all $f \in \mathcal{M}_{uq}^{\varphi^{1/u}}(\mathbb{R}^n)$.

Proof. Although the proof is simple, we include it for reader's convenience. We calculate that

$$\begin{aligned} \||f|^u\|_{\mathcal{M}_q^\varphi} &= \sup_{y \in \mathbb{R}^n, r > 0} \varphi(r) \left(\frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f(y)|^{uq} dy \right)^{\frac{1}{q}} \\ &= \sup_{y \in \mathbb{R}^n, r > 0} \left(\varphi(r)^{\frac{1}{u}} \left(\frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f(y)|^{uq} dy \right)^{\frac{1}{qu}} \right)^u \\ &= \left(\|f\|_{\mathcal{M}_{uq}^{\varphi^{1/u}}} \right)^u, \end{aligned}$$

as was to be shown. \square

In addition to general facts above, we need to exclude some special case where $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ is close to $L^\infty(\mathbb{R}^n)$. We invoke the following proposition from [39, Lemma 2].

Proposition 2.6. *If a nonnegative locally integrable function φ and a positive constant $C > 0$ satisfy*

$$(2.2) \quad \int_r^\infty \frac{ds}{\varphi(s)s} \leq \frac{C}{\varphi(r)} \quad (r > 0),$$

then

$$(2.3) \quad \int_r^\infty \frac{ds}{\varphi(s)s^{1-\varepsilon}} \leq \frac{C}{1-C\varepsilon} \cdot \frac{r^\varepsilon}{\varphi(r)} \quad (r > 0)$$

for all $0 < \varepsilon < C^{-1}$.

When we consider the vector-valued inequalities, the following observation will be useful.

Proposition 2.7. *Let φ be a nonnegative locally integrable function such that there exists a constant $C > 0$ such that $\varphi(s) \leq C\varphi(r)$ for all $r, s > 0$ with $\frac{1}{2} \leq \frac{r}{s} \leq 2$. Then the following are equivalent:*

- (1) φ satisfies (2.2).
- (2) φ satisfies (2.3) for some $\varepsilon > 0$.
- (3) There exist constants $\varepsilon > 0$ and $C > 0$ such that

$$(2.4) \quad \frac{t^\varepsilon}{\varphi(t)} \leq \frac{Cr^\varepsilon}{\varphi(r)} \quad (t \geq r)$$

If (1)–(3) are satisfied, then

$$(2.5) \quad \int_r^\infty \frac{ds}{\varphi(s)^u s} \leq \frac{C}{\varphi(r)^u} \quad (r > 0)$$

for all $0 < u < \infty$, where C depends only on u .

Proof. The implication (1) \implies (2) follows from Proposition 2.6.

Assume (2). Then we have

$$\frac{t^\varepsilon}{\varphi(t)} \lesssim \int_t^{2t} \frac{dv}{v^{1-\varepsilon}\varphi(v)} \lesssim \frac{r^\varepsilon}{\varphi(r)}$$

thanks to the doubling property of φ , proving (3).

If we assume (3), then we have

$$\int_r^\infty \frac{ds}{\varphi(s)s} = \int_r^\infty \frac{s^\varepsilon}{\varphi(s)} \frac{ds}{s^{1+\varepsilon}} \lesssim \int_r^\infty \frac{r^\varepsilon}{\varphi(r)} \frac{ds}{s^{1+\varepsilon}} = \frac{1}{\varepsilon\varphi(r)},$$

which implies (1). Note that (3) also implies (2.5) because φ^u satisfies (3) as well. \square

Remark 2.8. Inequality (2.5) is known to be necessary for (2.2); just remark that one can apply Proposition 2.7 to φ^u .

2.2. Vector-valued maximal inequality for $\mathcal{M}_q^\varphi(\mathbb{R}^n)$. Here we prove the following vector-valued inequality:

Theorem 2.9. *We denote by M the Hardy–Littlewood maximal operator defined by*

$$(2.6) \quad Mf(x) \equiv \sup_Q \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| dy$$

for $f \in L_{\text{loc}}^1(\mathbb{R}^n)$, where the supremum is taken over all cubes Q .

Let $1 < q < \infty$, $1 < r \leq \infty$ and $\varphi : (0, \infty) \rightarrow (0, \infty)$ be a function.

- (1) For a measurable function $f : \mathbb{R}^n \rightarrow \mathbb{C}$, we have

$$(2.7) \quad \|Mf\|_{\mathcal{M}_q^\varphi} \lesssim \|f\|_{\mathcal{M}_q^\varphi}.$$

In particular, for any sequence $\{f_j\}_{j=1}^\infty$ of $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ -functions,

$$(2.8) \quad \left\| \sup_{j \in \mathbb{N}} Mf_j \right\|_{\mathcal{M}_q^\varphi} \lesssim \left\| \sup_{j \in \mathbb{N}} |f_j| \right\|_{\mathcal{M}_q^\varphi}$$

(2) Assume (2.4). Then for any sequence $\{f_j\}_{j=1}^\infty$ of $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ -functions,

$$(2.9) \quad \left\| \left(\sum_{j=1}^\infty M f_j^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi} \lesssim \left\| \left(\sum_{j=1}^\infty |f_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi}.$$

Proof.

(1) See [47, Theorem 2.3] for (2.7). Note that (2.8) is a direct consequence of (2.7) and

$$\sup_{j \in \mathbb{N}} M f_j(x) \leq M \left[\sup_{j \in \mathbb{N}} |f_j| \right](x) \quad (x \in \mathbb{R}^n).$$

(2) See [47, Theorem 2.5] for (2.9). □

Next, we recall the following fundamental estimate:

Lemma 2.10. *Let R be a cube. Then*

$$(2.10) \quad M[\chi_R](x) \sim \frac{|R|}{|R| + |x - c(R)|^n} \quad (x \in \mathbb{R}^n),$$

where the implicit constants in (2.10) depend only on n .

Proof. The proof is standard. We content ourselves with its outline. For the proof, we shall distinguish two cases.

(1) $x \in 3R$. In this case, we can show that

$$\frac{1}{3^n} \leq M[\chi_R] \leq 1, \quad 1 \leq 1 + \frac{|x - c(R)|}{\ell(R)} \leq 1 + 3n.$$

(2) $x \in 3^{l+1}R \setminus 3^l R$ for some $l \in \mathbb{N}$. In this case, we can show that

$$\frac{1}{3^{(l+1)n}} \leq M[\chi_R](x) \leq \frac{4^n}{3^{ln}}, \quad \frac{3^l}{2} \leq 1 + \frac{|x - c(R)|}{\ell(R)} \leq 1 + n \cdot 3^{l+1}.$$

□

We present a function of $f \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$.

Proposition 2.11. *Let $1 \leq q < \infty$ and $\varphi \in \mathcal{G}_q$. Define*

$$f \equiv \sum_{j=-\infty}^\infty \frac{\chi_{[2^{-j-1}, 2^{-j}]^n}}{\varphi(2^{-j})}, \quad g \equiv \sum_{j=-\infty}^\infty \frac{\chi_{[0, 2^{-j}]^n}}{\varphi(2^{-j})}, \quad h \equiv \sup_{j \in \mathbb{Z}} \frac{\chi_{[0, 2^{-j}]^n}}{\varphi(2^{-j})}.$$

Define a decreasing function φ^\dagger by: $\varphi^\dagger(t) \equiv \varphi(t)t^{-n/q}$ for $t > 0$.

(1) Then the following are equivalent;

- (a) $f \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$,
- (b) $h \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$,
- (c) φ^\dagger satisfies the integral condition, or equivalently, there exists a constant $C > 0$

$$\text{such that } \sum_{j=-\infty}^l \frac{1}{\varphi^\dagger(2^{-j})} \leq \frac{C}{\varphi^\dagger(2^{-l})} \text{ for all } l \in \mathbb{Z}.$$

- (2) There exists a constant $C > 0$ such that $\sum_{j=l}^{\infty} \frac{1}{\varphi(2^{-j})} \leq \frac{C}{\varphi(2^{-l})}$ and that $\sum_{j=\infty}^l \frac{1}{\varphi^\dagger(2^{-j})} \leq \frac{C}{\varphi^\dagger(2^{-l})}$ for all $l \in \mathbb{Z}$ if and only if $g \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$.

Proof.

- (1) Note that $f \leq h \leq 2^n Mf$, where M denotes the Hardy-Littlewood maximal operator given by (2.6). Observe that M is bounded on $\mathcal{M}_q^\varphi(\mathbb{R}^n)$. Thus, (a) and (b) are equivalent. Since f is expressed as $f = f_0(\|\cdot\|_\infty)$, that is, there exists a function $f_0 : [0, \infty) \rightarrow \mathbb{R}$ such that $f(x) = f_0(\|x\|_\infty)$ for all $x \in \mathbb{R}^n$, where $\|\cdot\|_\infty$ denotes the ℓ^∞ -norm, it follows that (a) and (c) are equivalent.
- (2) Suppose first $g \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$. Then

$$\varphi(2^{-l}) \left(\frac{1}{|[0, 2^{-l}]^n|} \int_{[0, 2^{-l}]^n} \left(\sum_{j=l}^{\infty} \frac{1}{\varphi(2^{-j})} \right)^q dx \right)^{\frac{1}{q}} \leq \|g\|_{\mathcal{M}_q^\varphi}.$$

Thus, $\sum_{j=l}^{\infty} \frac{1}{\varphi(2^{-j})} \leq \frac{\|g\|_{\mathcal{M}_q^\varphi}}{\varphi(2^{-l})}$ for all $l \in \mathbb{Z}$. This implies that $f(x) \leq h(x) \leq g(x) \lesssim f(x)$.

Thus, from (1), $\sum_{j=\infty}^l \frac{1}{\varphi^\dagger(2^{-j})} \leq \frac{C}{\varphi^\dagger(2^{-l})}$ holds as well.

Conversely, assume that

$$(2.11) \quad \sum_{j=\infty}^l \frac{1}{\varphi^\dagger(2^{-j})} \leq \frac{C}{\varphi^\dagger(2^{-l})}$$

and

$$(2.12) \quad \sum_{j=l}^{\infty} \frac{1}{\varphi(2^{-j})} \leq \frac{C}{\varphi(2^{-l})}$$

hold for all $l \in \mathbb{Z}$. Then we have $g \sim f$ from (2.12). Thus, $f \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$ by (2.11), from which it follows that $g \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$.

□

Let $0 < \eta < \infty$. We define the powered Hardy-Littlewood maximal operator $M^{(\eta)}$ by

$$(2.13) \quad M^{(\eta)}f(x) \equiv \sup_{R>0} \left(\frac{1}{|Q(x, R)|} \int_{Q(x, R)} |f(y)|^\eta dy \right)^{\frac{1}{\eta}} \quad (x \in \mathbb{R}^n)$$

for a measurable function f . When we consider the atomic decomposition, we use the following observation:

Lemma 2.12. [8, Lemma A.2] *Let $\kappa \geq n$ and $\varepsilon > 0$. Then,*

$$(2.14) \quad \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \langle 2^\nu x - m \rangle^{-\kappa-\varepsilon} \right| \lesssim_\varepsilon M^{(\frac{n}{\kappa})} \left[\sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}} \right] (x).$$

Here, $M^{(\frac{n}{\kappa})}$ denotes is the powered Hardy-Littlewood maximal operator with $\eta \equiv \frac{n}{\kappa}$.

Remark 2.13. The vector-valued inequality is a key ingredient throughout the present paper. Probably it is easier to handle Herz spaces (see Section 7.3 for the definition) or Musielak-Orlicz spaces (see Section 7.5 for the definition) than Morrey spaces. In fact, Herz spaces and Musielak-Orlicz spaces have $L_c^\infty(\mathbb{R}^n)$ as a dense subspace.

2.3. A Hardy type inequality. We will need the following Hardy type inequality for later consideration:

Proposition 2.14. *Let $0 < r \leq \infty$ and $\delta > 0$. Then for all nonnegative sequences $\{A_j\}_{j=1}^\infty$,*

$$(2.15) \quad \left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} 2^{-|j-k|\delta} A_j \right)^r \right)^{\frac{1}{r}} \lesssim \left(\sum_{k=1}^{\infty} A_k^r \right)^{\frac{1}{r}}.$$

If $r = \infty$, (2.15) reads ;

$$(2.16) \quad \sup_{k \in \mathbb{N}} \left(\sum_{j=1}^{\infty} 2^{-|j-k|\delta} A_j \right) \lesssim \sup_{k \in \mathbb{N}} A_k.$$

Proof. The inequality (2.15) is proved in [8, Lemma A.2.1], while the inequality (2.16) is known as the discrete Hardy inequality. \square

2.4. A convolution estimate. We will make use of the following estimate on integrals. We define $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Lemma 2.15. [13, p.466] *Let $\nu, \mu \in \mathbb{Z}$ with $\nu \geq \mu$, $M > 0$ and $L \in \mathbb{N}_0$, and $N > M + L + n$. Suppose that a $C^L(\mathbb{R}^n)$ -function φ and x_φ are such that*

$$(2.17) \quad |\nabla^L \varphi(x)| \leq \frac{2^{\mu(n+L)}}{(1 + 2^\mu |x - x_\varphi|)^M}$$

for all $x \in \mathbb{R}^n$. Assume, in addition, that ψ is a measurable function such that

$$\int_{\mathbb{R}^n} x^\beta \psi(x) dx = 0, \quad \text{if } |\beta| \leq L - 1$$

and that, for some $x_\psi \in \mathbb{R}^n$,

$$|\psi(x)| \leq \frac{2^{\nu n}}{(1 + 2^\nu |x - x_\psi|)^N}$$

for all $x \in \mathbb{R}^n$. Then

$$\left| \int_{\mathbb{R}^n} \varphi(x) \psi(x) dx \right| \lesssim \frac{2^{\mu n - (\nu - \mu)L}}{(1 + 2^\mu |x_\varphi - x_\psi|)^M}.$$

Here are examples of applications of Lemma 2.15.

Example 2.16. Let $\Theta, f \in \mathcal{S}(\mathbb{R}^n)$. By using Lemma 2.15 with

$$\varphi = \mathcal{F}^{-1}[\Theta(2^{-k} \cdot)](-\cdot), \quad \psi = f, \quad x_\varphi = 0, \quad x_\psi = x,$$

and

$$L = \mu = 0, \quad \nu = k, \quad M = N_0, \quad N = [1 + N_0],$$

where N_0 is a constant obtained in Corollary 2.3, we obtain

$$|\Theta_k(D)f(x)| \lesssim p_{[1+N_0]}(f)(1 + |x|)^{-N_0}.$$

In fact, for example, we can check (2.17) as follows:

$$\begin{aligned} |\nabla^L \varphi(x)| &= |\nabla^L [2^{kn} \mathcal{F}^{-1} \tau(-2^k \cdot)](x)| \\ &= 2^{kn+kL} |\nabla^L [\mathcal{F}^{-1} \tau](-2^k x)| \\ &\lesssim 2^{kn+kL} p_{[1+N_0]}(\mathcal{F}^{-1} \tau) (1 + 2^k |x|)^{-N_0}. \end{aligned}$$

2.5. Plancherel-Polya-Nikol'skii inequality. Let Ω be a compact subset of \mathbb{R}^n . Recall that $\mathcal{S}_\Omega(\mathbb{R}^n)$ denotes the space of all elements $\varphi \in \mathcal{S}(\mathbb{R}^n)$ which satisfies $\text{supp } \mathcal{F}\varphi \subset \Omega$.

The following inequality will be used throughout the present paper.

Theorem 2.17 (Plancherel-Polya-Nikol'skii inequality, [69, Theorem 1.3.1, Section 1.4.1]). *Let $\eta > 0$ and φ be a function in $\mathcal{S}_{Q(1)}$. Then we have*

$$\sup_{z \in \mathbb{R}^n} \frac{|\varphi(x-z)|}{(1+|z|)^{\frac{\eta}{n}}} \lesssim_\eta M^{(\eta)} \varphi(x),$$

where $M^{(\eta)}$ is the maximal function given by (2.13). In particular, for any $R, \eta > 0$ and $\varphi \in \mathcal{S}_{Q(R)}$, we have the pointwise estimate

$$\sup_{z \in \mathbb{R}^n} \frac{|\varphi(x-z)|}{(1+R|z|)^{\frac{\eta}{n}}} \lesssim_\eta M^{(\eta)} \varphi(x).$$

Here we recall a typical application of the above theorem.

Example 2.18. Let Θ be a function supported in $Q(2)$. Define $\Theta_j(\xi) \equiv \Theta(2^{-j}\xi)$ for $j \in \mathbb{N}_0$. Then let us prove

$$(2.18) \quad |\Theta_j(D)f(x)| \lesssim \frac{1}{\varphi(2^{-j})} \|\Theta_j(D)f\|_{\mathcal{M}_q^\varphi}.$$

In fact, for any points x and y satisfying $|x-y| \leq 2^{-j}$ we have

$$(2.19) \quad |\Theta_j(D)f(x)| \lesssim \sup_{z \in \mathbb{R}^n} \frac{|\Theta_j(D)f(z)|}{(1+2^j|z-y|)^{2n/q}} \lesssim M^{(q/2)}[\Theta_j(D)f](y).$$

Therefore, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} |\Theta_j(D)f(x)| &\leq \left(2^{j(n+1)} \int_{Q(x, 2^{-j})} M^{(q/2)}[\Theta_j(D)f](y) dy \right)^{\frac{1}{q}} \\ &= \frac{1}{\varphi(2^{-j})} \cdot \varphi(2^{-j}) \left(2^{j(n+1)} \int_{Q(x, 2^{-j})} M^{(q/2)}[\Theta_j(D)f](y) dy \right)^{\frac{1}{q}} \end{aligned}$$

thanks to (1.1). By using the Morrey norm and Lemma 2.5, we obtain

$$\begin{aligned} |\Theta_j(D)f(x)| &\leq \frac{1}{\varphi(2^{-j})} \cdot \|M^{(q/2)}[\Theta_j(D)f]\|_{\mathcal{M}_q^\varphi} \\ &\leq \frac{1}{\varphi(2^{-j})} \cdot (\|M[|\Theta_j(D)f|^{q/2}]\|_{\mathcal{M}_2^{\varphi^{q/2}}})^{2/q} \\ &\lesssim \frac{1}{\varphi(2^{-j})} \cdot (\|\Theta_j(D)f\|_{\mathcal{M}_2^{\varphi^{q/2}}})^{2/q} \\ &= \|\Theta_j(D)f\|_{\mathcal{M}_q^\varphi}, \end{aligned}$$

which proves (2.18).

The following result is a consequence of the maximal inequalities in Theorem 2.9 and the Plancherel-Polya-Nikol'skii inequality.

Theorem 2.19 (Multiplier result). *Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$ and let*

$$\nu > \frac{n}{\min(1, q, r)} + \frac{n}{2}.$$

(1) *The following inequality is true:*

$$(2.20) \quad \|2^{js} H_{(j)}(D) f_j^* \|_{\mathcal{M}_q^\varphi} \lesssim \left(\sup_{k \in \mathbb{N}} \|H_{(k)}(d_k \cdot)\|_{H_2^\nu} \right) \|2^{js} f_j\|_{\mathcal{M}_q^\varphi}$$

holds for any $j \in \mathbb{N}$.

(2) *Assume (2.4) in addition. Suppose that, for each $j = 1, 2, \dots$, we are given a compact set K_j of \mathbb{R}^n with diameter d_j , $H_{(j)} \in H_2^\nu(\mathbb{R}^n)$ and $f_j \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$ such that $\text{supp}(\mathcal{F}f_j) \subseteq K_j$. Define*

$$H_{(j)}(D) f_j^*(x) \equiv \sup_{z \in \mathbb{R}^n} \frac{|\mathcal{F}^{-1}[H_{(j)} \mathcal{F}f_j](x - z)|}{(1 + d_j |z|)^{n/\eta}},$$

where $\eta \equiv \min(1, q, r)/2$. If a collection $\{f_j\}_{j=1}^\infty$ of measurable functions satisfies

$$\left\| \left(\sum_{j=1}^\infty 2^{jsr} |f_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi} < \infty,$$

then we have

$$(2.21) \quad \left\| \left(\sum_{j=1}^\infty 2^{jsr} |H_{(j)}(D) f_j^*|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi} \lesssim \left(\sup_{k \in \mathbb{N}} \|H_{(k)}(d_k \cdot)\|_{H_2^\nu} \right) \left\| \left(\sum_{j=1}^\infty 2^{jsr} |f_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi}.$$

Proof. The heart of the matter is to prove (2.22) and (2.23) below. For the sake of the convenience for readers, we prove (2.22) and (2.23). Let δ satisfy $\nu = \frac{n}{\eta} + \frac{n+\delta}{2}$. By the definition of $H_{(j)}(D) f_j$, we see that

$$\begin{aligned} & |2^{js} \mathcal{F}^{-1}[H_{(j)} \mathcal{F}f_j](x - z)| \\ & \lesssim \int_{\mathbb{R}^n} 2^{js} \frac{|(\mathcal{F}^{-1} H_{(j)})(x - z - y)|}{(1 + d_j |x - y|)^{n/\eta}} |f_j(y)| (1 + d_j |x - y|)^{n/\eta} dy \\ & \lesssim \sup_{u \in \mathbb{R}^n} \frac{2^{js} |f_j(u)|}{(1 + d_j |x - u|)^{n/\eta}} \int_{\mathbb{R}^n} |(\mathcal{F}^{-1} H_{(j)})(x - z - y)| (1 + d_j |x - y|)^{n/\eta} dy. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \frac{|2^{js} \mathcal{F}^{-1}[H_{(j)} \mathcal{F}f_j](x - z)|}{(1 + d_j |z|)^{n/\eta}} \\ & \lesssim \sup_{u \in \mathbb{R}^n} \frac{2^{js} |f_j(u)|}{(1 + d_j |x - u|)^{n/\eta}} \int_{\mathbb{R}^n} |(\mathcal{F}^{-1} H_{(j)})(x - z - y)| \left(\frac{1 + d_j |x - y|}{1 + d_j |z|} \right)^{n/\eta} dy \\ & \lesssim \sup_{u \in \mathbb{R}^n} \frac{2^{js} |f_j(u)|}{(1 + d_j |x - u|)^{n/\eta}} \int_{\mathbb{R}^n} |(\mathcal{F}^{-1} H_{(j)})(x - z - y)| (1 + d_j |x - y - z|)^{n/\eta} dy, \end{aligned}$$

where we used $1 + d_j |x - y| \leq (1 + d_j |x - y - z|)(1 + d_j |z|)$. After we apply the Hölder inequality to the integrand as above, we use the chain rule of differentiation and $\mathcal{F}f(d_j \cdot) = d_j^{-n} \mathcal{F}[f(d_j^{-1} \cdot)]$ ($d_j > 0$), then we obtain

$$\frac{|2^{js} \mathcal{F}^{-1}[H_{(j)} \mathcal{F}f_j](x - z)|}{(1 + d_j |z|)^{n/\eta}} \lesssim \|H_{(j)}(d_j \cdot)\|_{H_2^\nu} \sup_{u \in \mathbb{R}^n} \frac{2^{js} |f_j(x - u)|}{(1 + d_j |u|)^{n/\eta}}.$$

If we combine this estimate with Plancherel-Polya-Nikol'skii inequality (Theorem 2.17), then we have

$$(2.22) \quad \frac{|2^{js} \mathcal{F}^{-1}[H_{(j)} \mathcal{F} f_j](x-z)|}{(1+d_j|z|)^{n/\eta}} \lesssim \left(\sup_{k \in \mathbb{N}} \|H_{(k)}(d_k \cdot)\|_{H_2^\nu} \right) M^{(\eta)}[2^{js} f_j](x)$$

and hence

$$(2.23) \quad \left(\sum_{j=1}^{\infty} 2^{jsr} \mathcal{F}^{-1}[H_{(j)} \mathcal{F} f_j]^*(x)^r \right)^{\frac{1}{r}} \lesssim \left(\sup_{k \in \mathbb{N}} \|H_{(k)}(d_k \cdot)\|_{H_2^\nu} \right) \left(\sum_{j=1}^{\infty} M^{(\eta)}[2^{js} f_j](x)^r \right)^{\frac{1}{r}}.$$

If we consider the $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ -norm,

$$\begin{aligned} & \left\| \left(\sum_{j=1}^{\infty} (2^{js} \mathcal{F}^{-1}[H_{(j)} \mathcal{F} f_j]^*)^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi} \\ & \lesssim \left(\sup_{k \in \mathbb{N}} \|H_{(k)}(d_k \cdot)\|_{H_2^\nu} \right) \left\| \left(\sum_{j=1}^{\infty} M^{(\eta)}[2^{js} f_j]^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi} \\ & \lesssim \left(\sup_{k \in \mathbb{N}} \|H_{(k)}(d_k \cdot)\|_{H_2^\nu} \right) \left(\left\| \left(\sum_{j=1}^{\infty} M[|2^{js} f_j|^\eta]^{r/\eta} \right)^{\eta/r} \right\|_{\mathcal{M}_{q/\eta}^{\varphi/\eta}} \right)^{1/\eta}. \end{aligned}$$

Hence by combining Lemma 2.5 and Theorem 2.9, we obtain the desired inequality (2.21). \square

2.6. Reproducing formula. Rychkov proved the following reproducing formula:

Proposition 2.20. *Suppose that $\varphi_0 \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi_0(x) dx \neq 0$. Set*

$$(2.24) \quad \varphi_j(x) \equiv 2^{jn} \varphi_0(2^j x) - 2^{(j-1)n} \varphi_0(2^{j-1} x) \quad (x \in \mathbb{R}^n)$$

for $j \in \mathbb{N}$. Let $L \in \mathbb{N}$. Then there exists $\psi_0 \in C_c^\infty(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} x^\alpha \psi_1(x) dx = 0$$

for all $|\alpha| \leq L$ and that

$$(2.25) \quad f = \sum_{j \in \mathbb{N}_0} \psi_j * \varphi_j * f \quad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$. Here

$$(2.26) \quad \psi_j(x) \equiv 2^{jn} \psi_0(2^j x) - 2^{(j-1)n} \psi_0(2^{j-1} x) \quad (x \in \mathbb{R}^n)$$

for $j \in \mathbb{N}$.

Proof. See [46]. \square

Before we go further, a couple of remarks may be in order.

Remark 2.21.

- (1) A rescaling argument allows us to assume that φ_0 and ψ_0 are supported in $[-1/4, 1/4]^n$.

- (2) Let $L_1 \in \mathbb{N}$ be an arbitrary number. In Proposition 2.20, we can assume that there exists Φ such that $\Delta^{L_1}\Phi \equiv \varphi_1$. As a result,

$$\int_{\mathbb{R}^n} x^\beta \varphi_1(x) dx = 0$$

for all β with $|\beta| \leq 2L_1 - 1$.

The next lemma explains how to construct atoms.

Lemma 2.22. *Let $\{\varphi_j\}_{j \in \mathbb{N}_0}$ and $\{\psi_j\}_{j \in \mathbb{N}_0}$ as above. Assume in addition that*

$$\int_{\mathbb{R}^n} x^\beta \varphi_1(x) dx = 0$$

for all β with $|\beta| \leq L$. Define

$$(2.27) \quad \gamma_{jm}(x) \equiv \int_{Q_{jm}} \varphi_j(x-y) \cdot f * \psi_j(y) dy \quad (x \in \mathbb{R}^n)$$

for $f \in \mathcal{S}'(\mathbb{R}^n)$ and $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then we have;

- (1) $\gamma_{jm} \in C^\infty(\mathbb{R}^n)$ for all $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$,
- (2) $\text{supp}(\gamma_{jm}) \subset 3Q_{jm}$ for all $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$,
- (3) $\int_{\mathbb{R}^n} x^\beta \gamma_{jm}(x) dx = 0$ for all $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq L$, $j \in \mathbb{N}$ and $m \in \mathbb{Z}^n$.

Proof. All the assertions are easy to check. For example, we can check (2) as follows:

$$\begin{aligned} \text{supp}(\gamma_{jm}) &\subset Q_{jm} + [-2^{-j-2}, 2^{-j-2}]^n \\ &= 2^{-j}m + [0, 2^{-j}]^n + [-2^{-j-2}, 2^{-j-2}]^n \\ &= 2^{-j}m + [-2^{-j-2}, 5 \cdot 2^{-j-2}]^n \\ &= 3Q_{jm} \end{aligned}$$

for all $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. □

3. GENERALIZED TRIEBEL-LIZORKIN-MORREY SPACES ON \mathbb{R}^n

3.1. Proof of Theorem 1.4. We start with a setup. We recall that θ and τ are compactly supported functions satisfying

$$0 \notin \text{supp}(\tau), \quad \theta(\xi) > 0 \text{ if } \xi \in Q(2), \quad \tau(\xi) > 0 \text{ if } \xi \in Q(2) \setminus Q(1).$$

Let $\tilde{\theta}$ and $\tilde{\tau}$ be compactly supported functions satisfying

$$0 \notin \text{supp}(\tilde{\tau}), \quad \tilde{\theta}(\xi) > 0 \text{ if } \xi \in Q(2), \quad \tilde{\tau}(\xi) > 0 \text{ if } \xi \in Q(2) \setminus Q(1).$$

We define

$$(3.1) \quad \tau_k(\xi) \equiv \tau(2^{-k}\xi),$$

and

$$(3.2) \quad \tilde{\tau}_k(\xi) \equiv \tilde{\tau}(2^{-k}\xi)$$

for $\xi \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

We define $\|f\|_{\mathcal{M}_{q,r}^s(\theta,\tau)} \equiv \|f\|_{\mathcal{M}_{q,r}^s}$ as in (1.3) and (1.4).

By the symmetry, it is sufficient to prove that

$$(3.3) \quad \|f\|_{\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\theta, \tau)} \lesssim \|f\|_{\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\tilde{\theta}, \tilde{\tau})}.$$

With the above setup in mind, we prove

$$(3.4) \quad \left(\sum_{j=1}^{\infty} 2^{(j+3)sr} \|\tau_{j+3}(D)f\|_{\mathcal{M}_q^\varphi}^r \right)^{\frac{1}{r}} \lesssim \left(\sum_{j=1}^{\infty} 2^{jsr} \|\tilde{\tau}_j(D)f\|_{\mathcal{M}_q^\varphi}^r \right)^{\frac{1}{r}}.$$

Once (3.4) is proved, we can prove

$$(3.5) \quad \left(\sum_{j=1}^3 2^{jsr} \|\tau_j(D)f\|_{\mathcal{M}_q^\varphi}^r \right)^{\frac{1}{r}} \lesssim \|\tilde{\theta}(D)f\|_{\mathcal{M}_q^\varphi} \left(\sum_{j=1}^{\infty} 2^{jsr} \|\tilde{\tau}_j(D)f\|_{\mathcal{M}_q^\varphi}^r \right)^{\frac{1}{r}}$$

similarly to (3.4).

We can prove (3.4) with the help of Theorem 2.19. If $j \geq 4$, we see that

$$\begin{aligned} \tau_j(D)f &= \mathcal{F}^{-1}[\tau_j \mathcal{F}f] \\ &= \mathcal{F}^{-1} \left[(\tilde{\tau}_{j-1} + \tilde{\tau}_j + \tilde{\tau}_{j+1}) \frac{\tau_j}{(\tilde{\tau}_{j-1} + \tilde{\tau}_j + \tilde{\tau}_{j+1})} \mathcal{F}f \right] \\ &= \frac{\tau_j}{(\tilde{\tau}_{j-1} + \tilde{\tau}_j + \tilde{\tau}_{j+1})} (D) \mathcal{F}^{-1}[(\tilde{\tau}_{j-1} + \tilde{\tau}_j + \tilde{\tau}_{j+1}) \mathcal{F}f] \\ &= \left(\frac{\tau}{\tilde{\tau}_{-1} + \tilde{\tau} + \tilde{\tau}_1} \right)_j (D) \mathcal{F}^{-1}[(\tilde{\tau}_{j-1} + \tilde{\tau}_j + \tilde{\tau}_{j+1}) \mathcal{F}f]. \end{aligned}$$

By Theorem 2.19, we have the desired assertion.

3.2. Fundamental properties. First, we note that the following $\min(1, q, r)$ -triangle inequality holds. The proof being standard, we omit the proof.

Lemma 3.1. *Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$. Assume (2.4) in the case when $\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) = \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ with $r < \infty$. Then*

$$(\|f_1 + f_2\|_{\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s})^{\min(1, q, r)} \leq (\|f_1\|_{\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s})^{\min(1, q, r)} + (\|f_2\|_{\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s})^{\min(1, q, r)}$$

The next proposition deals with the lifting property.

Proposition 3.2 (Lift operator, Lifting property). *Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$. Assume in addition that φ satisfies (2.4) when $r < \infty$ and $\mathcal{A} = \mathcal{E}$. Then*

$$(1 - \Delta)^{-M/2} : \mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \rightarrow \mathcal{A}_{\mathcal{M}_q^\varphi, r}^{s+M}(\mathbb{R}^n)$$

is an isomorphism.

Proof. This is a consequence of Theorem 2.19 or [30, Theorem 3.10]. □

Next, we verify the embedding properties.

Proposition 3.3. *Let $0 < q < \infty$, $0 < r_1, r_2 \leq \infty$, $s \in \mathbb{R}$, $\varepsilon > 0$ and $\varphi \in \mathcal{G}_q$. Then*

$$\mathcal{A}_{\mathcal{M}_q^\varphi, r_1}^s(\mathbb{R}^n) \hookrightarrow \mathcal{A}_{\mathcal{M}_q^\varphi, r_2}^{s-\varepsilon}(\mathbb{R}^n).$$

Proof. When $r_1 \leq r_2$, then it is easy to see that the desired inequality holds by $\ell^{r_1}(\mathbb{N}_0) \hookrightarrow \ell^{r_2}(\mathbb{N}_0)$. In the case of $r_1 > r_2$, we can prove the desired inequality by using the same argument of Besov and Triebel-Lizorkin spaces situations. So we omit the proof. \square

Next, we investigate the relation between $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$.

We begin with the following quantitative estimate:

Lemma 3.4. *Let $0 < q < \infty$, $0 < r \leq \infty$ and $\varphi \in \mathcal{G}_q$. Assume that $s > 0$ is such that*

$$(3.6) \quad \sum_{j=1}^{\infty} \frac{1}{2^{js}\varphi(2^{-j})} < \infty.$$

Then

$$\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \hookrightarrow B_{\infty 1}^0(\mathbb{R}^n).$$

In particular, for such s ,

$$\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \hookrightarrow B_{\infty 1}^0(\mathbb{R}^n) \hookrightarrow \text{BUC}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Proof. Let $f \in \mathcal{N}_{\mathcal{M}_q^\varphi, \infty}^s(\mathbb{R}^n)$. Then we have

$$|\tau_j(D)f(x)| \lesssim \frac{1}{\varphi(2^{-j})} \cdot \|f\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, \infty}^s}$$

for all $x \in \mathbb{R}^n$ thanks to Example 2.18. Likewise, we have

$$|\theta(D)f(x)| \lesssim \left(\int_{Q(x,1)} M[|\theta(D)f|^{q/2}](y)^2 dy \right)^{\frac{1}{q}} \lesssim \frac{1}{\varphi(1)} \cdot \|f\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, \infty}^s}.$$

Thus, we have $\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \hookrightarrow B_{\infty 1}^0(\mathbb{R}^n) \hookrightarrow \text{BUC}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. \square

Condition (3.6) is a natural one as the following remark implies:

Remark 3.5.

- (1) (3.6) is satisfied when $s > \frac{n}{q}$.
- (2) (3.6) is also necessary for $\mathcal{N}_{\mathcal{M}_q^\varphi, \infty}^s(\mathbb{R}^n) \hookrightarrow B_{\infty 1}^0(\mathbb{R}^n)$. In fact, if $\zeta \in \mathcal{S}$ is such that $\chi_{Q(1.6) \setminus Q(1.5)} \leq \zeta \leq \chi_{Q(1.7) \setminus Q(1.4)}$, then

$$f = \sum_{j=1}^{\infty} \frac{(\mathcal{F}^{-1}\zeta)(2^j \cdot)}{\varphi(2^{-j})} \in \mathcal{A}_{\mathcal{M}_q^\varphi, \infty}^s(\mathbb{R}^n),$$

since

$$\|f\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, \infty}^s} = \sup_{j \in \mathbb{N}} \left\| \frac{(\mathcal{F}^{-1}\zeta)(2^j \cdot)}{\varphi(2^{-j})} \right\|_{\mathcal{M}_q^\varphi} \lesssim \sup_{j \in \mathbb{N}} \left\| \frac{[M[\chi_{Q(2^{-j})}]]^{2/q}}{\varphi(2^{-j})} \right\|_{\mathcal{M}_q^\varphi} \lesssim 1.$$

Meanwhile, we have

$$\|f\|_{B_{\infty 1}^0} \sim \sum_{j=1}^{\infty} \frac{1}{\varphi(2^{-j})2^{js}}.$$

Proposition 3.6. *Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$. Assume in addition that φ satisfies (2.4) when $r < \infty$ and $\mathcal{A} = \mathcal{E}$. Then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$$

in the sense of continuous embeddings.

Proof. Let us prove $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. The key observation is (2.1). Once this is obtained, we can resort to [30, Theorem 3.17]. Here for the sake of convenience for readers we outline the proof of [30, Theorem 3.17] by adapting it to our setting.

Since

$$(3.7) \quad (1 - \Delta)^{-M/2} : \mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \rightarrow \mathcal{A}_{\mathcal{M}_q^\varphi, r}^{s+M}(\mathbb{R}^n)$$

is an isomorphism for all $M > 0$ by virtue of Proposition 3.2 and

$$(3.8) \quad \mathcal{N}_{\mathcal{M}_q^\varphi, \infty}^s(\mathbb{R}^n) \hookrightarrow \mathcal{A}_{\mathcal{M}_q^\varphi, r}^{s-\varepsilon}(\mathbb{R}^n).$$

in the sense of continuous embedding thanks to Proposition 3.3 for all $\varepsilon > 0$, we have only to prove

$$(3.9) \quad \mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{N}_{\mathcal{M}_q^\varphi, \infty}^s(\mathbb{R}^n)$$

for all $s \leq 0$. Indeed, combining (3.8) and (3.9), we obtain

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{A}_{\mathcal{M}_q^\varphi, r}^{s-\varepsilon}(\mathbb{R}^n).$$

By the use of (3.7), we have

$$\mathcal{S}(\mathbb{R}^n) = (1 - \Delta)^{-M/2}[\mathcal{S}(\mathbb{R}^n)] \hookrightarrow (1 - \Delta)^{-M/2}[\mathcal{A}_{\mathcal{M}_q^\varphi, r}^{s-\varepsilon}(\mathbb{R}^n)] = \mathcal{A}_{\mathcal{M}_q^\varphi, r}^{s+M-\varepsilon}(\mathbb{R}^n).$$

Thus, the matters are reduced to proving (3.9).

Let $f \in \mathcal{S}(\mathbb{R}^n)$ and let N_0 be a constant obtained in Corollary 2.3. Then according to Lemma 2.15, we have

$$|\theta(D)f(x)| + |\tau_k(D)f(x)| \lesssim p_{[1+N_0]}(f)(1 + |x|)^{-N_0},$$

where the implicit constant in \lesssim does not depend on k . Thus,

$$\begin{aligned} \|f\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, \infty}^s} &= \|\theta(D)f\|_{\mathcal{M}_q^\varphi} + \sup_{k \in \mathbb{N}} 2^{ks} \|\tau_k(D)f\|_{\mathcal{M}_q^\varphi} \\ &\lesssim p_{[1+N_0]}(f) \left(\|(1 + |\cdot|)^{-N_0}\|_{\mathcal{M}_q^\varphi} + \sup_{k \in \mathbb{N}} 2^{ks} \|(1 + |\cdot|)^{-N_0}\|_{\mathcal{M}_q^\varphi} \right) \\ &\lesssim p_{[1+N_0]}(f), \end{aligned}$$

which proves $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$.

Let us now prove $\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Since

$$(1 - \Delta)^{-M/2} : \mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \rightarrow \mathcal{A}_{\mathcal{M}_q^\varphi, r}^{s+M}(\mathbb{R}^n)$$

is an isomorphism for all $M > 0$ and

$$\mathcal{A}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \hookrightarrow \mathcal{N}_{\mathcal{M}_q^\varphi, \infty}^s(\mathbb{R}^n)$$

in the sense of continuous embedding, we have only to prove

$$\mathcal{N}_{\mathcal{M}_q^\varphi, \infty}^s(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$$

for $s \gg 1$, which is already done in Lemma 3.4. \square

4. DECOMPOSITIONS

4.1. Atomic decomposition. We consider the atomic decomposition.

Definition 4.1. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$.

- (1) The (nonhomogeneous) generalized Besov-Morrey sequence space $\mathbf{n}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ is the set of all doubly indexed sequences $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ for which the quasi-norm

$$\|\lambda\|_{\mathbf{n}_{\mathcal{M}_q^\varphi, r}^s} \equiv \begin{cases} \left(\sum_{j=0}^{\infty} 2^{jsr} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{jm} \chi_{Q_{jm}} \right\|_{\mathcal{M}_q^\varphi}^r \right)^{\frac{1}{r}} & (r < \infty), \\ \sup_{j \in \mathbb{N}_0} 2^{js} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{jm} \chi_{Q_{jm}} \right\|_{\mathcal{M}_q^\varphi} & (r = \infty) \end{cases}$$

is finite.

- (2) The (nonhomogeneous) generalized Triebel-Lizorkin-Morrey sequence space $\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ is the set of all $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ for which the quasi-norm

$$\|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s} \equiv \begin{cases} \left\| \left\{ \sum_{j=0}^{\infty} 2^{jsr} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}| \chi_{Q_{jm}} \right)^r \right\}^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi} & (r < \infty), \\ \left\| \sup_{j \in \mathbb{N}_0} 2^{js} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}| \chi_{Q_{jm}} \right) \right\|_{\mathcal{M}_q^\varphi} & (r = \infty) \end{cases}$$

is finite.

- (3) The space $\mathbf{a}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ denotes either $\mathbf{n}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ or $\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Assume (2.4) in the case when $\mathbf{a}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) = \mathbf{e}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ with $r < \infty$.

Definition 4.2. Let $L \in \mathbb{N}_0 \cup \{-1\}$ and $K \in \mathbb{N}_0$.

- (1) Let $m \in \mathbb{Z}^n$. A C^K -function $a : \mathbb{R}^n \rightarrow \mathbb{C}$ is said to be a (K, L) -atom supported near Q_{0m} , if

$$(4.1) \quad |\partial^\alpha a(x)| \leq \chi_{3Q_{0m}}(x)$$

for all α with $|\alpha| \leq K$.

- (2) Let $j = 1, 2, \dots$ and $m \in \mathbb{Z}^n$. A C^K -function $a : \mathbb{R}^n \rightarrow \mathbb{C}$ is said to be a (K, L) -atom supported near Q_{jm} , if

$$(4.2) \quad 2^{-j|\alpha|} |\partial^\alpha a(x)| \leq \chi_{3Q_{jm}}(x)$$

for all α with $|\alpha| \leq K$ and

$$(4.3) \quad \int_{\mathbb{R}^n} x^\beta a(x) dx = 0$$

for all β with $|\beta| \leq L$ when $L \geq 0$.

- (3) Denote by $\mathfrak{A} = \mathfrak{A}(\mathbb{R}^n)$ the set of all collections $\{a_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ of C^K -functions such that each a_{jm} is a (K, L) -atom supported near Q_{jm} .

Before we proceed further, a helpful remark may be in order.

Remark 4.3. The number 3 does not count in the above definition; any number d will do as long as $d > 1$.

Theorem 4.4. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$. Let also $L \in \mathbb{N}_0 \cup \{-1\}$ and $K \in \mathbb{N}_0$. Assume

$$(4.4) \quad K \geq [1 + s]_+, \quad L \geq \max(-1, [\sigma_q - s]),$$

where $\sigma_q \equiv n \left(\frac{1}{q} - 1 \right)_+$.

- (1) Let $f \in \mathcal{N}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then there exist a family $\{a_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathfrak{A}$ and a doubly indexed complex sequence $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{n}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ such that

$$(4.5) \quad f = \sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right) \quad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

and that

$$(4.6) \quad \|\lambda\|_{\mathbf{n}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|f\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s}.$$

- (2) Let $\{a_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathfrak{A}$ and $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{n}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then

$$f \equiv \sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right)$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and belongs to $\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Furthermore,

$$\|f\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|\lambda\|_{\mathbf{n}_{\mathcal{M}_q^\varphi, r}^s}.$$

Theorem 4.5. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi : (0, \infty) \rightarrow (0, \infty) \in \mathcal{G}_q$. Let also $L \in \mathbb{N}_0 \cup \{-1\}$ and $K \in \mathbb{N}_0$. Assume

$$K \geq [1 + s]_+, \quad L \geq \max(-1, [\sigma_{qr} - s]),$$

where $\sigma_{qr} \equiv \max(\sigma_q, \sigma_r)$.

- (1) Let $f \in \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then there exist a family $\{a_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathfrak{A}$ and a doubly indexed complex sequence $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{e}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ satisfying (4.5) and that

$$(4.7) \quad \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s}.$$

- (2) Let $\{a_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathfrak{A}$ and $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{e}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then

$$f \equiv \sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right)$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and belongs to $\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Furthermore,

$$(4.8) \quad \|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s}.$$

Theorem 4.4(1) and Theorem 4.5(1) are already obtained in [30, Theorem 10.15]. So, we concentrate on the proof of Theorem 4.4(2) and Theorem 4.5(2). The conditions on K and L are milder. To prove them, we invoke Lemma 2.15. Its direct corollary is:

Corollary 4.6. Let $P > 0$ be arbitrary. Let $K \in \mathbb{N}_0$ and $L \in \mathbb{N}_0 \cup \{-1\}$. Suppose that we are given an atom a_{jm} supported near Q_{jm} .

- (1) Let $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then

$$(4.9) \quad |\theta(D)a_{jm}(x)| \lesssim 2^{-j(L+1)} M[\chi_{Q_{0m}}](x)^{\frac{N}{n}}.$$

In particular,

$$\left| \theta(D) \left(\sum_{m \in \mathbb{Z}^n} a_{jm} \right) (x) \right| \lesssim 2^{-j(L+1)} \sum_{m \in \mathbb{Z}^n} M[\chi_{Q_{0m}}](x)^{\frac{N}{n}}.$$

(2) Let $\nu \in \mathbb{N}$, $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then

$$(4.10) \quad |\tau_\nu(D)a_{jm}(x)| \lesssim \begin{cases} 2^{-(\nu-j)K} M[\chi_{Q_{jm}}](x)^{\frac{N}{n}} & (\nu \geq j), \\ 2^{(\nu-j)(L+1+n-P)} M[\chi_{Q_{jm}}](x)^{\frac{N}{n}} & (\nu \leq j). \end{cases}$$

In particular, by letting

$$(4.11) \quad \delta \equiv \min(L+1+n-P+s, K-s),$$

we have

$$2^{\nu s} \left| \tau_\nu(D) \left[\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right] (x) \right| \lesssim 2^{-|\nu-j|\delta} \sum_{m \in \mathbb{Z}^n} M[2^{js} \lambda_{jm} \chi_{Q_{jm}}](x)^{\frac{N}{n}}.$$

Proof. (4.9) is simpler than (4.10); we concentrate on (4.10). Define $\Phi^\nu(x) \equiv 2^{\nu n} \mathcal{F}^{-1} \tau(2^\nu x)$ for $x \in \mathbb{R}^n$. Then we have

$$\tau_\nu(D)a_{jm}(x) = \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} \Phi^\nu(x-y) a_{jm}(y) dy.$$

Let $x \in \mathbb{R}^n$ be fixed with this in mind.

Let $\nu \geq j$. Then we have

$$(4.12) \quad \int_{\mathbb{R}^n} x^\alpha \Phi^\nu(x) dx = (2\pi)^{\frac{n}{2}} i^{|\alpha|} \partial^\alpha [\tau(2^{-\nu} \cdot)](0) = 0$$

for all multi-indexes α . We use (4.12) for all α whose length is less than or equal to $K-1$. We then obtain

$$|\tau_\nu(D)a_{jm}(x)| = \frac{1}{2^{jn} \sqrt{(2\pi)^n}} \left| \int_{\mathbb{R}^n} \Phi^\nu(x-y) \cdot 2^{jn} a_{jm}(y) dy \right| \lesssim \frac{2^{-(\nu-j)K}}{(1+2^j|x-2^{-j}m|)^P}$$

by letting $\varphi = 2^{jn} a_{jm}$, $\psi = \Phi^\nu(x - \cdot)$, $x_\varphi = 2^{-j}m$, $x_\psi = x$ in Lemma 2.15. In fact, we can check (2.17) as follows:

$$|\nabla^\alpha \varphi(x)| = 2^{jn} |\nabla^\alpha a_{jm}(x)| \lesssim 2^{j(K+n)} \chi_{3Q_{jm}}(x) \lesssim \frac{2^{j(n+K)}}{(1+2^j|x-x_\varphi|)^P}.$$

By using Lemma 2.10, we obtain (4.10) for the case when $\nu \geq j$.

Let $\nu < j$. Then we have

$$|\partial_y^\alpha [\Phi^\nu(x-y)]| \lesssim_\alpha \frac{2^{(n+|\alpha|)\nu}}{(1+2^\nu|x-y|)^P}$$

for all multi-indexes α as well as (4.2) with $\alpha = 0$ and (4.3). Notice that our assumption $\nu < j$ excludes the case when $j = 0$; a_{jm} does satisfy (4.3). Thus,

$$|\tau_\nu(D)a_{jm}(x)| \lesssim \frac{2^{(\nu-j)(L+n+1)}}{(1+2^\nu|x-2^{-j}m|)^P} \quad (x \in \mathbb{R}^n)$$

and hence by using $j > \nu$ and Lemma 2.10 again,

$$|\tau_\nu(D)a_{jm}(x)| \lesssim \frac{2^{(\nu-j)(L+n+1)}}{(2^{\nu-j} + 2^\nu|x-2^{-j}m|)^P} \lesssim 2^{(\nu-j)(L+1+n-P)} M[\chi_{Q_{jm}}](x)^{\frac{P}{n}} \quad (x \in \mathbb{R}^n).$$

Thus, the proof is complete. \square

Remark 4.7. Recall that $\mathcal{E}_{pqr}^s(\mathbb{R}^n)$ is a special case of generalized Triebel-Lizorkin-Morrey spaces; see Section 7.2. The key ingredient is a counterpart of Theorems 4.4 and 4.5 to Morrey spaces. See Proposition 7.2 below.

Proof of Theorem 4.4(2) and Theorem 4.5(2). We prove Theorem 4.5(2), the proof of Theorem 4.4(2) being similar.

We choose real numbers P and P' so that

$$(4.13) \quad \frac{n}{\min(1, q, r)} < P' < P < L + n + 1 + s.$$

Then δ given by (4.11) is positive.

Let $\{a_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathfrak{A}$ and $\{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{e}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$.

Let us suppose for the time being that there exists $N \gg 1$ such that $\lambda_{jm} = 0$ if $j \geq N$. This implies $f \equiv \sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right)$ converges in $\mathcal{S}'(\mathbb{R}^n)$.

We calculate that

$$\sum_{j=0}^{\infty} \left| \theta(D) \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right) (x) \right| \lesssim 2^{-j(L+n+1+s-P)} \sum_{m \in \mathbb{Z}^n} 2^{js} |\lambda_{jm}| M[\chi_{Q_{jm}}](x)^{\frac{P}{n}}$$

and that

$$\begin{aligned} & \left\{ \sum_{k=1}^{\infty} \left(2^{ks} \sum_{j=0}^{\infty} \left| \tau_k(D) \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right) (x) \right| \right)^r \right\}^{\frac{1}{r}} \\ & \lesssim \left\{ \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \left(2^{-|j-k|\delta} \sum_{m \in \mathbb{Z}^n} 2^{js} |\lambda_{jm}| M[\chi_{Q_{jm}}](x)^{\frac{P}{n}} \right)^r \right\}^{\frac{1}{r}} \\ & \lesssim \left\{ \sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} 2^{js} |\lambda_{jm}| M[\chi_{Q_{jm}}](x)^{\frac{P}{n}} \right)^r \right\}^{\frac{1}{r}}. \end{aligned}$$

If we invoke Lemma 2.12, then we have

$$\begin{aligned} & \left\{ \sum_{k=1}^{\infty} \left(2^{ks} \sum_{j=0}^{\infty} \left| \varphi_k(D) \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right) (x) \right| \right)^r \right\}^{\frac{1}{r}} \\ & \lesssim \left\{ \sum_{j=0}^{\infty} \left(M \left[\left| \sum_{m \in \mathbb{Z}^n} 2^{js} \lambda_{jm} \chi_{Q_{jm}} \right|^{\frac{n}{P'}} \right] (x)^{\frac{P'}{n}} \right)^r \right\}^{\frac{1}{r}}. \end{aligned}$$

Thus,

$$\begin{aligned} \|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} & \lesssim \left\| \left\{ \sum_{j=0}^{\infty} \left(M \left[\left| \sum_{m \in \mathbb{Z}^n} 2^{js} \lambda_{jm} \chi_{Q_{jm}} \right|^{\frac{n}{P'}} \right] (x)^{\frac{P'}{n}} \right)^r \right\}^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi} \\ & \lesssim \left\| \left\{ \left(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{js} |\lambda_{jm}| \chi_{Q_{jm}} \right)^r \right\}^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi} \\ & = \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s}. \end{aligned}$$

This shows that Theorem 4.5(2) is proved for λ satisfying that there exists $N \gg 1$ such that $\lambda_{jm} = 0$ if $j \geq N$.

Let us remove this assumption. To this end, we set

$$f_j \equiv \sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm}.$$

Choose $\rho > 0$ so that

$$L \geq \max(-1, [\sigma_{qr} - s + \rho]),$$

where $\sigma_{qr} \equiv \max(\sigma_q, \sigma_r)$. Then according to what we have proved, we have

$$\|f_j\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^{s-\rho}} \lesssim 2^{-\rho j} \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s}.$$

Therefore, $f = \sum_{j=1}^\infty f_j$ converges in $\mathcal{E}_{\mathcal{M}_q^\varphi, r}^{s-\rho}(\mathbb{R}^n)$ and hence $\mathcal{S}'(\mathbb{R}^n)$. Again according to what we have proved, we also have

$$\left\| \sum_{j=1}^N f_j \right\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s}$$

with the constant independent of N . As a result, by letting $N \rightarrow \infty$, the Fatou property of $\mathcal{M}_q^\varphi(\mathbb{R}^n)$ yields $f \in \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ with

$$(4.14) \quad \|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s}.$$

4.2. Molecular decomposition. In analogy with the atomic decomposition, we can develop a theory of molecular decomposition as well.

Definition 4.8. Let $L \in \mathbb{N}_0 \cup \{-1\}$ and $K, N \in \mathbb{N}_0$ be such that $N > K + n$.

- (1) Let $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. A C^K -function $b : \mathbb{R}^n \rightarrow \mathbb{C}$ is said to be a (K, L, N) -molecule supported near Q_{jm} , if

$$\left| \frac{\partial^\alpha b}{\partial x^\alpha}(x) \right| \leq 2^{|\alpha|j} (1 + |2^j x - m|)^{-N}$$

with $|\alpha| \leq K$ and (4.3) with $|\beta| \leq L$ and $j \geq 1$ hold. When $L = -1$, it is understood that (4.3) is a void condition.

- (2) Denote by $\mathfrak{M} = \mathfrak{M}(\mathbb{R}^n)$ the set of all collections $\{b_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ of C^K -functions such that each b_{jm} is a (K, L, N) -molecule supported near Q_{jm} .

Theorem 4.9. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$. Let also $L \in \mathbb{N}_0 \cup \{-1\}$ and $K \in \mathbb{N}_0$. Assume

$$(4.15) \quad K \geq [1 + s]_+, \quad L \geq \max(-1, [\sigma_q - s]),$$

where $\sigma_q \equiv n \left(\frac{1}{q} - 1 \right)_+$.

- (1) Let $f \in \mathcal{N}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then there exist a family $\{b_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathfrak{M}$ and a doubly indexed complex sequence $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{n}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ such that

$$(4.16) \quad f = \sum_{j=0}^\infty \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} b_{jm} \right) \quad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

and that

$$(4.17) \quad \|\lambda\|_{\mathbf{n}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|f\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s}.$$

(2) Let $\{b_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathfrak{M}$ and $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{n}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then

$$f \equiv \sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} b_{jm} \right)$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and belongs to $\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Furthermore,

$$\|f\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|\lambda\|_{\mathbf{n}_{\mathcal{M}_q^\varphi, r}^s}.$$

Theorem 4.10. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi : (0, \infty) \rightarrow (0, \infty) \in \mathcal{G}_q$. Let also $L \in \mathbb{N}_0 \cup \{-1\}$ and $K \in \mathbb{N}_0$. Assume

$$K \geq [1 + s]_+, \quad L \geq \max(-1, [\sigma_{qr} - s]),$$

where $\sigma_{qr} \equiv \max(\sigma_q, \sigma_r)$.

(1) Let $f \in \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then there exist a family $\{b_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathfrak{M}$ and a doubly indexed complex sequence $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{e}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ satisfying (4.16) and that

$$(4.18) \quad \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s}.$$

(2) Let $\{b_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathfrak{M}$ and $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{e}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then

$$f \equiv \sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} b_{jm} \right)$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and belongs to $\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Furthermore,

$$\|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s}.$$

We do not prove Theorem 4.9(1) and Theorem 4.10(1); Theorem 4.4(1) and Theorem 4.5(1) are stronger assertions than Theorem 4.9(1) and Theorem 4.10(1), respectively. We concentrate on the proof of Theorem 4.10(2); that of Theorem 4.9(2) is similar.

Proof of Theorem 4.10(2). We modify Corollary 4.6 as follows:

Corollary 4.11. Let $K, N \in \mathbb{N}_0$ and $L \in \mathbb{N}_0 \cup \{-1\}$ with $N > K + n$. Suppose that we are given an molecule b_{jm} supported near Q_{jm} .

(1) Let $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then

$$(4.19) \quad |\theta(D)b_{jm}(x)| \lesssim 2^{-j(L+1)} M[\chi_{Q_{0m}}](x)^{\frac{p}{n}}$$

In particular,

$$\left| \theta(D) \left(\sum_{m \in \mathbb{Z}^n} b_{jm} \right) (x) \right| \lesssim 2^{-j(L+1)} \sum_{m \in \mathbb{Z}^n} M[\chi_{Q_{0m}}](x)^{\frac{N}{n}}.$$

(2) Let $\nu \in \mathbb{N}$, $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then

$$(4.20) \quad |\tau_\nu(D)b_{jm}(x)| \lesssim \begin{cases} 2^{-(\nu-j)K} M[\chi_{Q_{jm}}](x)^{\frac{N}{n}} & (\nu \geq j), \\ 2^{(\nu-j)(L+1+n-N)} M[\chi_{Q_{jm}}](x)^{\frac{N}{n}} & (\nu \leq j). \end{cases}$$

In particular, by letting

$$(4.21) \quad \delta \equiv \min(L + 1 + n - N + s, K - s),$$

we have

$$2^{\nu s} \left| \tau_\nu(D) \left[\sum_{m \in \mathbb{Z}^n} \lambda_{jm} b_{jm} \right] (x) \right| \lesssim 2^{-|\nu-j|\delta} \sum_{m \in \mathbb{Z}^n} M[2^{js} \lambda_{jm} \chi_{Q_{jm}}](x)^{\frac{N}{n}}.$$

The proof is the same as that for Corollary 4.6.

The proof of Theorem 4.10(2) is a modification of the corresponding assertions. Since we are assuming $L \geq [\sigma_{qr} - s]$, we have $\frac{n}{\min(1, q, r)} < L + n + 1 + s$. Let \tilde{N} be a real number slightly less than $L + n + 1 + s$. By considering $\min(N, \tilde{N})$, we can assume that $N < L + n + 1 + s$. Choose $P' > 0$ so that

$$(4.22) \quad \frac{n}{\min(1, q, r)} < P' < N < L + n + 1 + s.$$

Note that conditions (4.13) and (4.22) are the same if we let $P = N$. Therefore, we can go through the same argument as we did in Theorem 4.5(2).

4.3. Quarkonial decomposition. By using the atomic decomposition, we can consider the quarkonial decomposition. All the results in this section are new; the quarkonial decomposition was not obtained in [30].

Definition 4.12 (ψ for the quarkonial decomposition). Throughout this section, the function $\psi \in \mathcal{S}$ is fixed so that $\{\psi(\cdot - m)\}_{m \in \mathbb{Z}^n}$ forms a *partition of unity*;

$$(4.23) \quad \sum_{m \in \mathbb{Z}^n} \psi(\cdot - m) \equiv 1.$$

Accordingly, choose $R > 0$ so that

$$(4.24) \quad \text{supp}(\psi) \subset Q(2^R).$$

With ψ specified as above, we define quarks.

Definition 4.13 (Regular quark). Let $\beta \in \mathbb{N}_0^n$, $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then define a function ψ^β and the quark $(\beta qu)_{\nu m} = (\beta qu)_{\nu, m}$ by:

$$(4.25) \quad \psi^\beta(x) \equiv x^\beta \psi(x), (\beta qu)_{\nu m}(x) \equiv \psi^\beta(2^\nu x - m) = (2^\nu x - m)^\beta \psi(2^\nu x - m)$$

for $x \in \mathbb{R}^n$. Each $(\beta qu)_{\nu m}$ is called the *quark*.

Remark 4.14. As it is mentioned in [70, Discussion 2.5, p12], there exists $d > 0$ such that

$$(4.26) \quad \text{supp}(\beta qu)_{\nu m} \subset dQ_{\nu m}$$

for any $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$.

By (4.24), we have $|\psi^\beta(x)| \leq 2^{R|\beta|}$ for any $\beta \in \mathbb{N}_0^n$. Therefore we obtain

$$|(\beta qu)_{\nu m}| \leq 2^{R|\beta|} \quad \text{with } \beta \in \mathbb{N}_0^n.$$

Fix any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq K$ and $K \geq 0$. Then, by the definition of quarks, we see that

$$|\partial^\alpha \psi_\beta(x)| = |\partial^\alpha [x^\beta \psi(x)]| \leq \sum_{|\alpha'| \leq K} |\partial^{\alpha'} [x^\beta \psi(x)]| \leq c_1(1 + |\beta|)^K 2^{R|\beta|} \leq c_2 2^{(R+\epsilon)|\beta|}$$

for any $\epsilon > 0$, where c_1 and c_2 are constants independent of β but depend on ψ , K and ϵ . This and the chain rule of differentiation imply that

$$(4.27) \quad |\partial^\alpha (\beta qu)_{\nu m}(x)| \lesssim 2^{\nu|\alpha| + (R+\epsilon)|\beta|}$$

holds for any $\beta \in \mathbb{N}_0^n$, $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$.

Lemma 4.15. Let $\{\Lambda_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ be a bounded sequence and let $\kappa \in \mathcal{S}$ satisfy $\chi_{Q(3)} \leq \kappa \leq \chi_{Q(3+1/100)}$. Then

$$\sum_{m \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{-\rho|\beta|}}{\beta!} \Lambda_{\nu m} \partial^\beta \mathcal{F}^{-1} \kappa(2^{-\rho} l - m) \psi^\beta(2^{\nu+\rho} \cdot - m)$$

is convergent in the weak star topology of $L^\infty(\mathbb{R}^n)$. More precisely, by writing

$$M \equiv \sup_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} |\Lambda_{\nu m}|,$$

we have

$$\int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{-\rho|\beta|}}{\beta!} |\Lambda_{\nu m} \partial^\beta \mathcal{F}^{-1} \kappa(2^{-\rho} l - m) \psi^\beta(2^{\nu+\rho} x - m) f(x)| dx \lesssim M \|f\|_{L^1}$$

for $f \in L^1(\mathbb{R}^n)$.

Proof. It suffices to prove that

$$(4.28) \quad \sum_{m \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{-\rho|\beta|}}{\beta!} |\Lambda_{\nu m} \partial^\beta \mathcal{F}^{-1} \kappa(2^{-\rho} l - m) \psi^\beta(2^{\nu+\rho} \cdot - m)| \in L^\infty(\mathbb{R}^n)$$

thanks to the Lebesgue convergence theorem.

Note that

$$|\partial^\beta \mathcal{F}^{-1} \kappa(2^{-\rho} l - m)| \lesssim \langle \beta \rangle^{2N} \langle 2^{-\rho} l - m \rangle^{-2N} \text{ and that } |\psi^\beta(2^{\nu+\rho} \cdot - m)| \lesssim 2^{(R+\varepsilon)|\beta|}.$$

For each $x \in \mathbb{R}^n$, let us fix $m_{x,1}, \dots, m_{x,N(R)}$ such that $\psi^\beta(2^{\nu+\rho} x - m) \neq 0$, where $N(R)$ is a geometric constant depending only on R . Then $|m_{x,j} - 2^{\nu+\rho} x| \leq 2^R$ for each $j = 1, 2, \dots, N(R)$.

We calculate that:

$$\begin{aligned} & \sum_{m \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{-\rho|\beta|}}{\beta!} |\Lambda_{\nu m} \partial^\beta \mathcal{F}^{-1} \kappa(2^{-\rho} l - m) \psi^\beta(2^{\nu+\rho} x - m)| \\ & \lesssim M \sum_{j=1}^{N(R)} \sum_{l \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{(R+\varepsilon-\rho)|\beta|}}{\beta!} \langle \beta \rangle^{2N} \langle 2^{-\rho} l - m_{x,j} \rangle^{-2N} \\ & \lesssim M \sum_{l \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{(R+\varepsilon-\rho)|\beta|}}{\beta!} \langle \beta \rangle^{2N} \langle 2^{-\rho} l - 2^{\nu+\rho} x \rangle^{-2N} \\ & \lesssim M \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{(R+\varepsilon-\rho)|\beta|}}{\beta!} \langle \beta \rangle^{2N} \int_{\mathbb{R}^n} \langle z - 2^{\nu+\rho} x \rangle^{-2N} dz \sim M \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{(R+\varepsilon-\rho)|\beta|}}{\beta!} \langle \beta \rangle^{2N} \sim M. \end{aligned}$$

Thus, (4.28) is obtained. \square

Definition 4.16 (Sequence spaces for quarkonial decomposition). Let R, ρ satisfy (4.24) and $\rho > R$. For a triply indexed complex sequence $\lambda = \{\lambda_{\nu m}^\beta\}_{\beta \in \mathbb{N}_0^n, \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$, define

$$(4.29) \quad \lambda^\beta \equiv \{\lambda_{\nu m}^\beta\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}, \quad \|\lambda\|_{\mathbf{a}_{\mathcal{M}_q^s, r, \rho}} \equiv \sup_{\beta \in \mathbb{N}_0^n} 2^{\rho|\beta|} \|\lambda^\beta\|_{\mathbf{a}_{\mathcal{M}_q^s, r}^\varphi}.$$

Theorem 4.17. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$. Assume $\rho \equiv [R+1] > R$, where R is a constant in (4.24).

(1) Let $s > \sigma_q$ and $f \in \mathcal{N}_{\mathcal{M}_q^s, r}^s(\mathbb{R}^n)$. Then there exists a triply indexed complex sequence

$$\lambda = \{\lambda_{\nu m}^\beta\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n}$$

such that

$$(4.30) \quad f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta q u)_{\nu m}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and

$$(4.31) \quad \|\lambda\|_{\mathbf{n}_{\mathcal{M}_q^s, r, \rho}} \lesssim \|f\|_{\mathcal{N}_{\mathcal{M}_q^s, r}^s}.$$

The constant $\lambda_{\nu m}^\beta$ depends continuously and linearly on f .

(2) If $s > \sigma_q$ and $\lambda = \{\lambda_{\nu m}^\beta\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n}$ satisfies $\|\lambda\|_{\mathbf{n}\mathcal{M}_{q,r}^s} < \infty$, then

$$(4.32) \quad f \equiv \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta q u)_{\nu m}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and belongs to $\mathcal{N}_{\mathcal{M}_{q,r}^s}(\mathbb{R}^n)$. Furthermore,

$$(4.33) \quad \|f\|_{\mathcal{N}_{\mathcal{M}_{q,r}^s}} \lesssim \|\lambda\|_{\mathbf{n}\mathcal{M}_{q,r}^s}.$$

(3) If $s > \sigma_{qr}$ and $f \in \mathcal{E}_{\mathcal{M}_{q,r}^s}(\mathbb{R}^n)$, then there exists a triply indexed complex sequence $\lambda = \{\lambda_{\nu m}^\beta\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n}$ such that

$$(4.34) \quad f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta q u)_{\nu m}$$

in $\mathcal{S}'(\mathbb{R}^n)$ and

$$(4.35) \quad \|\lambda\|_{\mathbf{e}\mathcal{M}_{q,r}^s} \lesssim \|f\|_{\mathcal{E}_{\mathcal{M}_{q,r}^s}}.$$

The constant $\lambda_{\nu m}^\beta$ depends continuously and linearly on f .

(4) If $s > \sigma_{qr}$ and $\lambda = \{\lambda_{\nu m}^\beta\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n}$ satisfies $\|\lambda\|_{\mathbf{e}\mathcal{M}_{q,r}^s} < \infty$, then

$$f \equiv \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta q u)_{\nu m}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and belongs to $\mathcal{E}_{\mathcal{M}_{q,r}^s}(\mathbb{R}^n)$. Furthermore,

$$\|f\|_{\mathcal{E}_{\mathcal{M}_{q,r}^s}} \lesssim \|\lambda\|_{\mathbf{e}\mathcal{M}_{q,r}^s}.$$

To prove Theorem 4.17, we need several Lemmas.

From now on we assume that θ and τ both belong to $\mathcal{S}(\mathbb{R}^n)$ that

$$\chi_{Q(2)} \leq \theta \leq \chi_{Q(3)},$$

and that

$$\tau_j = \theta(2^{-j} \cdot) - \theta(2^{-j+1} \cdot)$$

for $j \in \mathbb{N}$.

Lemma 4.18. Let $\kappa \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\chi_{Q(3)} \leq \kappa \leq \chi_{Q(3+1/100)}$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$.

(1) [53, Theorem 5.1.22] Whenever $\text{supp}(\mathcal{F}f) \subset Q(3 \cdot 2^\nu)$, f can be written as:

$$(4.36) \quad f = \frac{1}{\sqrt{(2\pi)^n}} \sum_{m \in \mathbb{Z}^n} f(2^{-\nu} m) \mathcal{F}^{-1} \kappa(2^\nu \cdot - m).$$

(2) [53, Corollary 5.1.23] Generally,

$$(4.37) \quad \begin{aligned} f &= \frac{1}{\sqrt{(2\pi)^n}} \sum_{m \in \mathbb{Z}^n} \tau(D) f(m) \mathcal{F}^{-1} \kappa(\cdot - m) \\ &+ \frac{1}{\sqrt{(2\pi)^n}} \sum_{\nu=1}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \varphi_\nu(D) f(2^{-\nu} m) \mathcal{F}^{-1} \kappa(2^\nu \cdot - m) \right). \end{aligned}$$

Lemma 4.19. Let $l \in \mathbb{Z}^n$, $0 < \eta < \min(1, q, r)$. Then

$$(4.38) \quad \|\lambda^l\|_{\mathbf{a}\mathcal{M}_{q,r}^s} \lesssim \langle l \rangle^{n/\eta} \|\lambda\|_{\mathbf{a}\mathcal{M}_{q,r}^s}.$$

Proof. We can prove this Lemma by using the same argument of the proof of [53, Lemma 5.1.24]. So we omit the proof. \square

Lemma 4.20. [53, Lemma 5.1.25] *Let $\kappa \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\chi_{Q(3)} < \kappa < \chi_{Q(3+1/100)}$. Then*

$$(4.39) \quad |\partial^\alpha \mathcal{F}^{-1} \kappa(y)| \lesssim_N \langle \alpha \rangle^{2N} \langle y \rangle^{-2N} \quad (\alpha \in \mathbb{N}_0^n, \quad y \in \mathbb{R}^n)$$

hold for any $N \gg 1$.

Proof of Theorem 4.17. Firstly, we prove (2) and (4). We let

$$\eta_0 \equiv \min(q, r, 1).$$

Assuming $\rho > R$, we can take $\epsilon > 0$ such that $0 < \epsilon < \rho - R$. By the assumption $s > \sigma_q$ and $s > \sigma_{qr}$ in (2) and (4) respectively, the atoms in $\mathcal{A}_{\mathcal{M}_q^s, r}^s(\mathbb{R}^n)$ are not required to satisfy any moment conditions. This and Remark 4.14 imply that we can regard $2^{-(R+\epsilon)|\beta|}(\beta qu)_{\nu m}$ as a $(K, -1)$ -atom supported near $Q_{\nu m}$ (modulo a multiplicative constant independent of λ and $\{a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$).

We define

$$(4.40) \quad f^\beta \equiv \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta qu)_{\nu m}$$

for each $\beta \in \mathbb{N}_0^n$.

By using Theorems 4.4 and 4.5, we have

$$\|f^\beta\|_{\mathcal{A}_{\mathcal{M}_q^s, r}^s} \lesssim \|2^{(R+\epsilon)|\beta|} \lambda^\beta\|_{\mathbf{a}_{\mathcal{M}_q^s, r}^s} = 2^{-(\rho-R-\epsilon)|\beta|} \|2^{\rho|\beta|} \lambda^\beta\|_{\mathbf{a}_{\mathcal{M}_q^s, r}^s} \lesssim 2^{-(\rho-R-\epsilon)|\beta|} \|\lambda\|_{\mathbf{a}_{\mathcal{M}_q^s, r, \rho}^s}.$$

Therefore, by the η_0 -triangle inequality (see Lemma 3.1), we see that

$$(\|f\|_{\mathcal{A}_{\mathcal{M}_q^s, r}^s})^{\eta_0} = \left(\left\| \sum_{\beta \in \mathbb{N}_0^n} f^\beta \right\|_{\mathcal{A}_{\mathcal{M}_q^s, r}^s} \right)^{\eta_0} \leq \sum_{\beta \in \mathbb{N}_0^n} (\|f^\beta\|_{\mathcal{A}_{\mathcal{M}_q^s, r}^s})^{\eta_0} \lesssim (\|\lambda\|_{\mathbf{a}_{\mathcal{M}_q^s, r, \rho}^s})^{\eta_0}.$$

This implies that (2) and (4) hold.

Finally, we prove (1) and (3). Let $f \in \mathcal{A}_{\mathcal{M}_q^s, r}^s(\mathbb{R}^n)$. By Lemma 4.18 we have

$$(4.41) \quad \begin{aligned} f &= \frac{1}{\sqrt{(2\pi)^n}} \sum_{m \in \mathbb{Z}^n} \theta(D) f(m) \mathcal{F}^{-1} \kappa(\cdot - m) \\ &+ \frac{1}{\sqrt{(2\pi)^n}} \sum_{\nu \in \mathbb{N}} \left(\sum_{m \in \mathbb{Z}^n} \tau_\nu(D) f(2^{-\nu} m) \mathcal{F}^{-1} \kappa(2^\nu \cdot - m) \right). \end{aligned}$$

We put

$$(4.42) \quad \Lambda_{\nu m} \equiv \begin{cases} \theta(D) f(m) & \text{if } \nu = 0, \\ \tau_\nu(D) f(2^{-\nu} m) & \text{if } \nu \neq 0. \end{cases}$$

Then we can rewrite (4.41) as

$$(4.43) \quad f \sim_n \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \Lambda_{\nu m} \mathcal{F}^{-1} \kappa(2^\nu \cdot - m),$$

where the symbol $A \sim_n B$ denotes that there exists a constant $c_n \neq 0$ such that $A = c_n B$. Since we may consider that ρ is a big integer, we can use the Taylor expansion to $\mathcal{F}^{-1} \kappa(2^\nu \cdot - m)$

at $x = 2^{-\nu-\rho}l$. Therefore we see that

$$\begin{aligned} \psi(2^{\nu+\rho}x - l)\mathcal{F}^{-1}\kappa(2^\nu x - m) &= \sum_{\beta \in \mathbb{N}_0^n} \frac{\partial^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l - m)(2^\nu x - 2^{-\rho}l)^\beta \psi(2^{\nu+\rho}x - l)}{\beta!} \\ &= \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{-\rho|\beta|} \partial^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l - m)(\beta qu)_{\nu+\rho,l}(x)}{\beta!}. \end{aligned}$$

Furthermore we have

$$\begin{aligned} \tau_\nu(D)f &= \frac{1}{\sqrt{(2\pi)^n}} \sum_{m \in \mathbb{Z}^n} \tau_\nu(D)f(2^{-\nu}m) \mathcal{F}^{-1}\kappa(2^\nu \cdot - m) \\ (4.44) \quad &\sim_n \sum_{m \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{-\rho|\beta|}}{\beta!} \Lambda_{\nu m} \partial^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l - m)(\beta qu)_{\nu+\rho,l} \end{aligned}$$

by (4.23). Since the convergence of (4.44) takes place also in the weak-* topology of $L^\infty(\mathbb{R}^n)$, we can change the order of summation in (4.44) as follows:

$$\begin{aligned} \tau_\nu(D)f &\sim_n \sum_{l \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \sum_{m \in \mathbb{Z}^n} \frac{2^{-\rho|\beta|}}{\beta!} \Lambda_{\nu m} \partial^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l - m)(\beta qu)_{\nu+\rho,l} \\ (4.45) \quad &\sim_n \sum_{l \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \lambda_{\nu+\rho,l}^\beta (\beta qu)_{\nu+\rho,l}, \end{aligned}$$

where $\lambda_{\nu+\rho,l}^\beta \equiv \frac{2^{-\rho|\beta|}}{\beta!} \sum_{m \in \mathbb{Z}^n} \Lambda_{\nu m} \partial^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l - m)$.

Let l_0 be a lattice point in $[0, 2^\rho)^n$ and $x \in Q_{\nu+\rho, 2^\rho l+l_0}$. Then we obtain

$$|\lambda_{\nu+\rho, 2^\rho l+l_0}^\beta| \lesssim 2^{-\rho|\beta|} \sum_{m \in \mathbb{Z}^n} \langle l - m \rangle^{-N} |\Lambda_{\nu m}| = 2^{-\rho|\beta|} \sum_{m \in \mathbb{Z}^n} \langle m \rangle^{-N} |\Lambda_{\nu, m+l}|$$

by Lemma 4.20. Put

$$(4.46) \quad \Lambda^m \equiv \{|\Lambda_{\nu, m+l}|\}_{\nu \in \mathbb{N}_0, l \in \mathbb{Z}^n} \quad (m \in \mathbb{Z}^n).$$

Then we have

$$\|\lambda^\beta\|_{\mathbf{a}_{\mathcal{M}_q^s, r}} \lesssim 2^{-\rho|\beta|} \left\| \sum_{m \in \mathbb{Z}^n} \langle m \rangle^{-N} |\Lambda^m| \right\|_{\mathbf{a}_{\mathcal{M}_q^s, r}} \lesssim 2^{-\rho|\beta|} \left(\sum_{m \in \mathbb{Z}^n} \langle m \rangle^{-N\eta_0} (\|\Lambda^m\|_{\mathbf{a}_{\mathcal{M}_q^s, r}}^s)^{\eta_0} \right)^{1/\eta_0}.$$

Since we can take N sufficiently large, by Lemma 4.19 with $\eta = \eta_0/2$, we see that

$$(4.47) \quad \|\lambda^\beta\|_{\mathbf{a}_{\mathcal{M}_q^s, r}} \lesssim 2^{-\rho|\beta|} \left(\sum_{m \in \mathbb{Z}^n} \langle m \rangle^{(2n/\eta_0 - N)\eta_0} (\|\Lambda\|_{\mathbf{a}_{\mathcal{M}_q^s, r}}^s)^{\eta_0} \right)^{1/\eta_0} \sim 2^{-\rho|\beta|} \|\Lambda\|_{\mathbf{a}_{\mathcal{M}_q^s, r}}^s.$$

This implies $\|\lambda\|_{\mathbf{a}_{\mathcal{M}_q^s, r, \rho}} \lesssim \|\Lambda\|_{\mathbf{a}_{\mathcal{M}_q^s, r}}^s$.

For any $y \in Q_{\nu, m}$, by the Plancherel-Polya-Nikol'skii inequality (Theorem 2.17), we have

$$\frac{1}{(1 + 2^\nu |y - 2^{-\nu}m|)^{\frac{2n}{\eta_0}}} |\tau_\nu(D)f(2^{-\nu}m)| \lesssim M^{(\eta_0/2)} [\tau_\nu(D)f](y).$$

Hence we see that

$$\begin{aligned}
|\Lambda_{\nu,m}| &= |\tau_\nu(D)f(2^{-\nu}m)| \\
&= (1 + 2^\nu|y - 2^{-\nu}m|)^{\frac{2n}{\eta_0}} \cdot \frac{1}{(1 + 2^\nu|y - 2^{-\nu}m|)^{\frac{2n}{\eta_0}}} |\tau_\nu(D)f(2^{-\nu}m)| \\
&\lesssim (1 + \sqrt{n})^{\frac{2n}{\eta_0}} \frac{1}{(1 + 2^\nu|y - 2^{-\nu}m|)^{\frac{2n}{\eta_0}}} |\tau_\nu(D)f(2^{-\nu}m)| \\
&\lesssim M^{(\eta_0/2)}[\tau_\nu(D)f](y).
\end{aligned}$$

Since we have

$$|\Lambda_{\nu,m}| \lesssim \inf_{y \in Q_{\nu,m}} \left(M^{(\eta_0/2)}[\tau_\nu(D)f](y) \right),$$

we obtain $\|\Lambda\|_{\mathcal{M}_{q,r}^s} \lesssim \|f\|_{\mathcal{A}_{q,r}^s}$. This proves the necessity of the quarkonial decomposition.

5. FUNDAMENTAL THEOREMS

5.1. Trace operator. In this section, we aim to extend the trace operator, which is initially defined on $\mathcal{S}(\mathbb{R}^n)$ by:

$$(5.1) \quad f \in \mathcal{S}(\mathbb{R}^n) \mapsto f(\cdot', 0_n) \in \mathcal{S}(\mathbb{R}^{n-1}).$$

Our main result is as follows:

Theorem 5.1. *Let $n \geq 2$. Suppose that we are given parameters q, r, s and a function $\varphi \in \mathcal{G}_q$. Define s^* and φ^* by*

$$(5.2) \quad s^* \equiv s - \frac{1}{q}$$

and

$$(5.3) \quad \varphi^*(t) \equiv \varphi(t)t^{-1/q} \quad (t > 0).$$

Assume in addition that φ^* is increasing and satisfies

$$(5.4) \quad \sum_{j=0}^{\infty} \frac{1}{\varphi^*(2^j s)} \lesssim \frac{1}{\varphi^*(s)} \quad (0 < s \leq 1).$$

(1) Let

$$(5.5) \quad s > \frac{1}{q} + (n-1) \left(\frac{1}{\min(1, q)} - 1 \right).$$

Then we can extend the trace operator $f \mapsto f(\cdot', 0_n)$ to a bounded linear operator from $\mathcal{N}_{\mathcal{M}_{q,r}^s}^s(\mathbb{R}^n)$ to $\mathcal{N}_{\mathcal{M}_{q,r}^{s^*}}^{s^*}(\mathbb{R}^{n-1})$.

(2) Let

$$(5.6) \quad s > \frac{1}{q} + (n-1) \left(\frac{1}{\min(1, q, r)} - 1 \right).$$

Then we can extend the trace operator $f \mapsto f(\cdot', 0_n)$ to a bounded linear operator from $\mathcal{E}_{\mathcal{M}_{q,r}^s}^s(\mathbb{R}^n)$ to $\mathcal{E}_{\mathcal{M}_{q,r}^{s^*}}^{s^*}(\mathbb{R}^{n-1})$.

Proof. We shall prove (2). The proof of (1) is even simpler. Let $Q' \in \mathcal{D}(\mathbb{R}^{n-1})$. We have only to prove

$$(5.7) \quad \varphi^*(\ell(Q')) \left(\frac{1}{|Q'|} \int_{Q'} \sum_{j=j_{Q'}}^{\infty} \left| 2^{js^*} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{j(m',0)} \chi_{Q_{jm'}}(x') \right|^q dx' \right)^{\frac{1}{q}} \lesssim \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^{\varphi,r}}^s}$$

and

$$(5.8) \quad \varphi^*(\ell(Q')) \left(\frac{1}{|Q'|} \int_{Q'} \sum_{j=0}^{j_{Q'}} \left| 2^{js^*} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{j(m',0)} \chi_{Q_{jm'}}(x') \right|^q dx' \right)^{\frac{1}{q}} \lesssim \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^{\varphi,r}}^s}$$

for all $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{e}_{\mathcal{M}_q^{\varphi,r}}^s(\mathbb{R}^n)$, where $j_{Q'} \equiv -\log_2 \ell(Q')$. Assuming (5.7) and (5.8) for the time being, let us conclude the proof.

For $f \in \mathcal{E}_{\mathcal{M}_q^{\varphi,r}}^s(\mathbb{R}^n)$, we use Theorem 4.4(2);

$$(5.9) \quad f = \sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right),$$

where $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{e}_{\mathcal{M}_q^{\varphi,r}}^s(\mathbb{R}^n)$ satisfies

$$\|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^{\varphi,r}}^s} \lesssim \|f\|_{\mathcal{E}_{\mathcal{M}_q^{\varphi,r}}^s}.$$

and $\{a_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathfrak{A}$. Define $\text{Tr} f$ by:

$$(5.10) \quad \text{Tr} f \equiv \sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm}(\cdot, 0_n) \right).$$

The definition of $\text{Tr} f$ makes sense; see Remark 5.2. Since we are assuming (5.6), we see

$$\{a_{j(m',0)}(\cdot, 0_n)\}_{j \in \mathbb{N}_0, m' \in \mathbb{Z}^{n-1}} \in \mathfrak{A}(\mathbb{R}^{n-1});$$

recall that no moment condition (4.3) is required because we can suppose $L = -1$.

Since each a_{jm} is supported in $\overline{3Q_{jm}} = 2^{-j}m + [-2^{-j}, 2^{j+1}]^n$, in order that $a_{jm}(\cdot, 0_n)$ is not a zero function, we need $2^{-j}m_n - 2^{-j} < 0 < 2^{-j}m_n + 2^{-j+1}$, or equivalently, $m_n = 0, -1$. Thus, we have

$$\text{Tr} f \equiv \sum_{j=0}^{\infty} \left(\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{j(m',0)} a_{j(m',0)}(\cdot, 0_n) \right) + \sum_{j=0}^{\infty} \left(\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{j(m',-1)} a_{j(m',-1)}(\cdot, 0_n) \right).$$

Let us set $\lambda' \equiv \{\lambda_{j(m',0)}\}_{j \in \mathbb{N}_0, m' \in \mathbb{Z}^{n-1}}$ and $\lambda^\dagger \equiv \{\lambda_{j(m',-1)}\}_{j \in \mathbb{N}_0, m' \in \mathbb{Z}^{n-1}}$. Combining (5.7) and (5.8), we have $\|\lambda'\|_{\mathbf{e}_{\mathcal{M}_q^{\varphi^*,q}}^{s^*}} \lesssim \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^{\varphi,r}}^s}$. By Theorem 2.9, we have a similar estimate for λ^\dagger :

$$\|\lambda^\dagger\|_{\mathbf{e}_{\mathcal{M}_q^{\varphi^*,q}}^{s^*}} \lesssim \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^{\varphi,r}}^s}. \text{ Hence}$$

$$\|\text{Tr} f\|_{\mathcal{E}_{\mathcal{M}_q^{\varphi^*,q}}^{s^*}} \lesssim \|\lambda'\|_{\mathbf{e}_{\mathcal{M}_q^{\varphi^*,q}}^{s^*}} + \|\lambda^\dagger\|_{\mathbf{e}_{\mathcal{M}_q^{\varphi^*,q}}^{s^*}} \lesssim \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^{\varphi,r}}^s} \lesssim \|f\|_{\mathcal{E}_{\mathcal{M}_q^{\varphi,r}}^s}.$$

Thus, the matters are reduced to (5.7) and (5.8).

To prove (5.7) and (5.8), let us set

$$\mathbb{I} \equiv \varphi^*(\ell(Q')) \left(\frac{1}{|Q'|} \int_{Q'} \sum_{j=j_{Q'}}^{\infty} \left| 2^{js^*} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{j(m',0)} \chi_{Q_{jm'}}(x') \right|^q dx' \right)^{\frac{1}{q}}$$

and

$$\Pi \equiv \varphi^*(\ell(Q')) \left(\frac{1}{|Q'|} \int_{Q'} \sum_{j=0}^{j_{Q'}} \left| 2^{js^*} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{j(m',0)} \chi_{Q_{jm'}}(x') \right|^q dx' \right)^{\frac{1}{q}}.$$

Let us start with simplifying (5.7). Note that

$$I = \varphi^*(\ell(Q')) \left(\frac{1}{|Q'|} \sum_{j=j_{Q'}}^{\infty} 2^{js^*q} \sum_{\substack{m' \in \mathbb{Z}^{n-1} \\ Q_{jm'} \subset Q'}} 2^{-j(n-1)} |\lambda_{j(m',0)}|^q \right)^{\frac{1}{q}}.$$

If we write

$$(5.11) \quad E(S) \equiv S \times (\ell(S), 2\ell(S))$$

and

$$(5.12) \quad F(S) \equiv S \times (0, 2\ell(S))$$

for $S \in \mathcal{D}(\mathbb{R}^{n-1})$, then $\{E(S)\}_{S \in \mathcal{D}(\mathbb{R}^{n-1})}$ is disjoint and hence

$$\begin{aligned} I &= 2^{\frac{1}{q}} \varphi(\ell(Q')) \left(\frac{1}{|F(Q')|} \int_{F(Q')} \sum_{j=j_{Q'}}^{\infty} 2^{jsq} \left| \sum_{\substack{m' \in \mathbb{Z}^{n-1} \\ Q_{jm'} \subset Q'}} \lambda_{j(m',0)} \chi_{E(Q_{jm'})}(x) \right|^q dx \right)^{\frac{1}{q}} \\ &= 2^{\frac{1}{q}} \varphi(\ell(Q')) \left\{ \frac{1}{|F(Q')|} \int_{F(Q')} \left(\sum_{j=j_{Q'}}^{\infty} 2^{jsr} \left| \sum_{\substack{m' \in \mathbb{Z}^{n-1} \\ Q_{jm'} \subset Q'}} \lambda_{j(m',0)} \chi_{E(Q_{jm'})}(x) \right|^r \right)^{\frac{q}{r}} dx \right\}^{\frac{1}{q}}. \end{aligned}$$

Observe that we have a pointwise estimate:

$$\left| \sum_{\substack{m' \in \mathbb{Z}^{n-1} \\ Q_{jm'} \subset Q'}} \lambda_{j(m',0)} \chi_{E(Q_{jm'})}(x) \right| \lesssim \left(M \left[\left| \sum_{\substack{m' \in \mathbb{Z}^{n-1} \\ Q_{jm'} \subset Q'}} \lambda_{j(m',0)} \chi_{Q_{j(m',0)}} \right|^u \right] (x) \right)^{\frac{1}{u}}$$

for all $0 < u < \infty$. Set $Q \equiv Q' \times [0, \ell(Q')]$ and $j_Q \equiv -\log_2 \ell(Q) = j_{Q'}$. By the Fefferman-Stein inequality for $L^q(\ell^r)$,

$$\begin{aligned}
\text{I} &\lesssim \varphi(\ell(Q')) \left\{ \frac{1}{|F(Q')|} \int_{\mathbb{R}^n} \left(\sum_{j=j_{Q'}}^{\infty} 2^{jsr} \left| \sum_{\substack{m' \in \mathbb{Z}^{n-1} \\ Q_{jm'} \subset Q'}} \lambda_{j(m',0)} \chi_{Q_{j(m',0)}}(x) \right|^r \right)^{\frac{q}{r}} dx \right\}^{\frac{1}{q}} \\
&\sim \varphi(\ell(Q)) \left\{ \frac{1}{|Q|} \int_Q \left(\sum_{j=j_{Q'}}^{\infty} 2^{jsr} \left| \sum_{\substack{m' \in \mathbb{Z}^{n-1} \\ Q_{jm'} \subset Q'}} \lambda_{j(m',0)} \chi_{Q_{j(m',0)}}(x) \right|^r \right)^{\frac{q}{r}} dx \right\}^{\frac{1}{q}} \\
&\leq \varphi(\ell(Q)) \left\{ \frac{1}{|Q|} \int_Q \left(\sum_{j=j_{Q'}}^{\infty} 2^{jsr} \left| \sum_{\substack{m' \in \mathbb{Z}^{n-1} \\ Q_{jm'} \subset Q'}} \lambda_{j(m',0)} \chi_{Q_{j(m',0)}}(x) \right|^r \right)^{\frac{q}{r}} dx \right\}^{\frac{1}{q}} \\
&\lesssim \|\lambda\|_{\mathbf{e}^s_{\mathcal{M}_q^{\varphi},r}},
\end{aligned}$$

which proves (5.7).

It remains to prove (5.8). For all j with $j < j_Q$, we can find a unique cube $Q_{jm'(j)} \in \mathcal{D}(\mathbb{R}^{n-1})$ such that $Q_{jm'(j)} \supset Q'$, where $m'(j) \in \mathbb{Z}^{n-1}$. Recall also s^* is defined by (5.2). Thus, the left-hand side of (5.8) simplifies to read;

$$\text{II} = \varphi^*(\ell(Q')) \left(\sum_{j=0}^{j_{Q'}} 2^{js^*q} |\lambda_{j(m'(j),0)}|^q \right)^{\frac{1}{q}} = \frac{\varphi(\ell(Q'))}{\ell(Q')^{1/q}} \left(\sum_{j=0}^{j_{Q'}} 2^{jsq-j} |\lambda_{j(m'(j),0)}|^q \right)^{\frac{1}{q}}.$$

For each $j = 0, 1, 2, \dots, j'_Q$, we write $Q_{(j)}$ to be the unique dyadic cube $R \in \mathcal{D}_j$ containing Q . By a trivial estimate $2^{js} |\lambda_{j(m'(j),0)}| \leq \varphi(2^{-j})^{-1} \|\lambda\|_{\mathbf{e}^s_{\mathcal{M}_q^{\varphi},r}}$ and (5.3), we obtain

$$\begin{aligned}
\text{II} &= \frac{\varphi(\ell(Q'))}{\ell(Q')^{1/q}} \left(\sum_{j=0}^{j_{Q'}} 2^{jsq-j} |\lambda_{j(m'(j),0)}|^q \right)^{\frac{1}{q}} \\
&= \frac{\varphi(\ell(Q))}{\ell(Q)^{1/q}} \left(\sum_{j=0}^{j_Q} \frac{2^{-j}}{|Q|} \int_Q 2^{jsq} |\lambda_{j(m'(j),0)}|^q \chi_{Q_{j(m'(j),0)}}(x) dx \right)^{\frac{1}{q}} \\
&\leq \frac{\varphi(\ell(Q))}{\ell(Q)^{1/q}} \left(\sum_{j=0}^{j_Q} \frac{2^{-j}}{|Q_{(j)}|} \int_{Q_{(j)}} \left| 2^{js} \sum_{m \in \mathbb{Z}^n} |\lambda_{jm}| \chi_{Q_{jm}}(x) \right|^q dx \right)^{\frac{1}{q}} \\
&\leq \frac{\varphi(\ell(Q))}{\ell(Q)^{1/q}} \left(\sum_{j=0}^{j_Q} \frac{2^{-j}}{\varphi(\ell(Q_{(j)}))^q} \right)^{\frac{1}{q}} \|\lambda\|_{\mathbf{e}^s_{\mathcal{M}_q^{\varphi},r}} = \frac{\varphi(\ell(Q))}{\ell(Q)^{1/q}} \left(\sum_{j=0}^{j_Q} \frac{\ell(Q_{(j)})}{\varphi(\ell(Q_{(j)}))^q} \right)^{\frac{1}{q}} \|\lambda\|_{\mathbf{e}^s_{\mathcal{M}_q^{\varphi},r}} \\
&\lesssim \|\lambda\|_{\mathbf{e}^s_{\mathcal{M}_q^{\varphi},r}},
\end{aligned}$$

where we used (5.4) for the last estimate. Thus, (5.8) is proved. \square

A helpful remark on the definition of the trace operator may be in order.

Remark 5.2. The trace operator defined by (5.10) is a linear operator which coincides with (5.1). According to the proof of Theorem 4.5(1), for each $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, there exists a continuous linear operator $I_{jm} : \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \rightarrow L_c^\infty(\mathbb{R}^n)$ such that $I_{jm}(f) = \lambda_{jm} a_{jm}$. Therefore, we can write

$$\text{Tr} f = \sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} I_{jm}(f)(\cdot, 0_n) \right).$$

Meanwhile if $f \in \mathcal{S}(\mathbb{R}^n)$, then the limit (5.9) takes place in $\text{BC}(\mathbb{R}^n)$ since $f \in B_{\infty\infty}^\varepsilon(\mathbb{R}^n) \hookrightarrow \text{BC}(\mathbb{R}^n)$. More precisely,

$$\lim_{J \rightarrow \infty} \left(\sup_{x \in \mathbb{R}^n} \left| f(x) - \sum_{j=0}^J \sum_{m \in \mathbb{Z}^n} I_{jm}(x) \right| \right) = 0.$$

Therefore, the convergence of (5.9) takes place pointwise, meaning that $\text{Tr} f$, defined by (5.10), agrees with the standard definition.

We discuss the surjectivity of the trace operator.

Theorem 5.3. *The trace operator defined in Theorem 5.1 in (1) and (2) is surjective.*

Proof. We shall prove that the trace operator defined in Theorem 5.1(2) is surjective, since the one defined in Theorem 5.1(1) can be proved surjective in a similar manner. To this end, it suffices to prove

$$(5.13) \quad \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|\lambda'\|_{\mathbf{e}_{\mathcal{M}_q^{\varphi^*, q}}^{s^*}},$$

where $\lambda' = \{\lambda'_{jm'}\}_{j \in \mathbb{N}_0, m' \in \mathbb{Z}^{n-1}}$ is a given doubly indexed complex sequence and we define a doubly indexed complex sequence by:

$$(5.14) \quad \lambda = \{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \equiv \{\delta_{m_n 0} \lambda'_{jm'}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n}.$$

Assuming (5.13) for a while, let us prove that any $f \in \mathcal{E}_{\mathcal{M}_q^{\varphi^*, q}}^{s^*}(\mathbb{R}^{n-1})$ is realized as $f = \text{Tr} g$ for some $g \in \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. By the use of Theorem 4.5(1), there exist $\lambda' = \{\lambda'_{jm'}\}_{j \in \mathbb{N}_0, m' \in \mathbb{Z}^{n-1}} \in \mathbf{e}_{\mathcal{M}_q^{\varphi^*, q}}^{s^*}(\mathbb{R}^{n-1})$ and $\{a'_{jm'}\}_{j \in \mathbb{N}_0, m' \in \mathbb{Z}^{n-1}} \in \mathfrak{A}(\mathbb{R}^{n-1})$ such that

$$f = \sum_{j=0}^{\infty} \left(\sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{jm'} a_{jm'} \right)$$

in $\mathcal{S}'(\mathbb{R}^{n-1})$. Choose a function $\Theta \in C^\infty(\mathbb{R})$ such that $\chi_{[-1/4, 1/4]} \leq \Theta \leq \chi_{[-1/2, 1/2]}$. Define λ by (5.14) and a function $A_{jm} \in C^K$ with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ by:

$$A_{jm}(x) = A_{jm}(x', x_n) = \begin{cases} a_{jm'}(x') \Theta(2^j x_n) & \text{if } m_n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let us write the left-hand side of (5.13) out in full by using an equivalent expression:

$$\|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s} \sim \sup_{Q \in \mathcal{D}(\mathbb{R}^n)} \varphi(\ell(Q)) \left\{ \frac{1}{|Q|} \int_Q \left(\sum_{j=0}^{\infty} 2^{jsr} \left| \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{Q_{j(m'), 0}}(x) \right|^r dx \right)^{\frac{q}{r}} \right\}^{\frac{1}{q}}.$$

Then we have

$$g \equiv \sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} A_{jm} \right) \in \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$$

from (5.13) and Theorem 4.5(2). Likewise, we have $\text{Trg} = f$ since $\Theta(0) = 1$. Thus, admitting (5.13), we can construct the desired g .

We need to prove

$$(5.15) \quad \varphi(\ell(Q)) \left\{ \frac{1}{|Q|} \int_Q \left(\sum_{j=0}^{\infty} \left| 2^{js} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{Q_{j(m',0)}}(x) \right|^r \right)^{\frac{q}{r}} dx \right\}^{\frac{1}{q}} \lesssim \|\lambda'\|_{\mathcal{M}_q^{s*,q}}^*$$

for the proof of (5.13). Let us write $G(Q') \equiv Q' \times (0, \ell(Q'))$ for $Q' \in \mathcal{D}(\mathbb{R}^{n-1})$. Suppose for a while that $Q \in \mathcal{D}(\mathbb{R}^n)$ is expressed as $Q = G(Q')$ for some $Q' \in \mathcal{D}(\mathbb{R}^{n-1})$. With this in mind, let us decompose estimate (5.15) into two parts:

$$(5.16) \quad \frac{\varphi(\ell(Q'))^q}{|G(Q')|} \int_{G(Q')} \left(\sum_{j=j_{Q'}}^{\infty} \left| 2^{js} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{Q_{j(m',0)}}(x) \right|^r \right)^{\frac{q}{r}} dx \lesssim (\|\lambda'\|_{\mathcal{M}_q^{s*,q}}^*)^q$$

and

$$(5.17) \quad \frac{\varphi(\ell(Q'))^q}{|G(Q')|} \int_{G(Q')} \left(\sum_{j=0}^{j_{Q'}} \left| 2^{js} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{Q_{j(m',0)}}(x) \right|^r \right)^{\frac{q}{r}} dx \lesssim (\|\lambda'\|_{\mathcal{M}_q^{s*,q}}^*)^q.$$

Let us prove (5.16). Set

$$\alpha \equiv \frac{\min(1, q, r)}{2}.$$

Let us also recall that $E(S)$ is given by (5.11). By the Fefferman-Stein inequality, we have

$$\begin{aligned} & \varphi(\ell(Q')) \left\{ \frac{1}{|G(Q')|} \int_{G(Q')} \left(\sum_{j=j_{Q'}}^{\infty} 2^{jsr} \left| \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{Q_{j(m',0)}}(x) \right|^r \right)^{\frac{q}{r}} dx \right\}^{\frac{1}{q}} \\ & \lesssim \varphi(\ell(Q')) \left\{ \frac{1}{|G(Q')|} \int_{G(Q')} \left(\sum_{j=j_{Q'}}^{\infty} 2^{jsr} M \left[\left| \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{E(Q_{jm'})} \right|^{\alpha} \right] (x)^{\frac{r}{\alpha}} \right)^{\frac{q}{r}} dx \right\}^{\frac{1}{q}} \\ & \lesssim \varphi(\ell(Q')) \left\{ \frac{1}{|G(Q')|} \int_{\mathbb{R}^n} \left(\sum_{j=j_{Q'}}^{\infty} 2^{jsr} M \left[\left| \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{E(Q_{jm'})} \right|^{\alpha} \right] (x)^{\frac{r}{\alpha}} \right)^{\frac{q}{r}} dx \right\}^{\frac{1}{q}} \\ & = \varphi(\ell(Q')) \left\{ \frac{1}{|G(Q')|} \int_{3G(Q')} \left(\sum_{j=j_{Q'}}^{\infty} 2^{jsr} \left| \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{E(Q_{jm'})}(x) \right|^r \right)^{\frac{q}{r}} dx \right\}^{\frac{1}{q}}. \end{aligned}$$

Note that, for any fixed $j \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, there exists at most only one $m \in \mathbb{Z}^n$ such that $\lambda'_{jm'} \chi_{E(Q_{jm'})}(x) \neq 0$. Thus,

$$\begin{aligned} & \varphi(\ell(Q')) \left\{ \frac{1}{|G(Q')|} \int_{3G(Q')} \left(\sum_{j=j_{Q'}}^{\infty} 2^{jsr} \left| \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{E(Q_{jm'})}(x) \right|^r \right)^{\frac{q}{r}} dx \right\}^{\frac{1}{q}} \\ &= \varphi(\ell(Q')) \left\{ \frac{1}{|G(Q')|} \int_{3G(Q')} \left(\sum_{j=j_{Q'}}^{\infty} 2^{jsq} \left| \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{E(Q_{jm'})}(x) \right|^q \right)^{\frac{q}{q}} dx \right\}^{\frac{1}{q}} \\ &= \varphi(\ell(Q'))^* \left\{ \frac{1}{|Q'|} \int_{Q'} \left(\sum_{j=j_{Q'}}^{\infty} 2^{jsq} \left| \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{Q_{jm'}}(x') \right|^q \right)^{\frac{q}{q}} dx' \right\}^{\frac{1}{q}} \leq \|\lambda'\|_{\mathcal{M}_q^{s*,q}}. \end{aligned}$$

This proves (5.16).

The proof of (5.17) is similar to (5.8). We omit the detail.

Finally, suppose that Q is not of the form $Q = G(Q')$ for some $Q' \in \mathcal{D}(\mathbb{R}^{n-1})$. This means that Q has a form as $Q = Q' \times [j\ell(Q'), (j+1)\ell(Q')]$ or $Q = Q' \times [-(j+1)\ell(Q'), -j\ell(Q')]$ for some $Q' \in \mathcal{D}(\mathbb{R}^{n-1})$ and $j \geq 1$. Due to symmetry let us suppose $Q = Q' \times [j\ell(Q'), (j+1)\ell(Q')]$. If $j\ell(Q') \geq 1$, then the left-hand side of (5.15) is zero and there is nothing to prove. Assume otherwise; $1 \leq j < \ell(Q')^{-1}$. Then by letting $w \equiv \frac{1}{2} \min(1, q, r)$, we have

$$\begin{aligned} & \varphi(\ell(Q)) \left\{ \frac{1}{|Q|} \int_Q \left(\sum_{j=0}^{\infty} 2^{js} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{Q_{j(m',0)}}(x) \right)^r dx \right\}^{\frac{1}{q}} \\ & \lesssim \varphi(\ell(Q)) \left\{ \frac{1}{|G(Q')|} \int_Q \left(\sum_{j=0}^{\infty} 2^{js} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{Q_{j(m',0)}}(x) \right)^r dx \right\}^{\frac{1}{q}} \\ & \lesssim \varphi(\ell(Q)) \left\{ \frac{1}{|G(Q')|} \int_Q \left(\sum_{j=0}^{\infty} 2^{js} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} M[\chi_{E(Q_{jm'}) \cap Q}](x) \right)^r dx \right\}^{\frac{1}{q}} \\ & \lesssim \varphi(\ell(Q)) \left\{ \frac{1}{|G(Q')|} \int_Q \left(\sum_{j=0}^{\infty} 2^{js} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{E(Q_{jm'}) \cap Q}(x) \right)^r dx \right\}^{\frac{1}{q}} \\ & \sim \varphi(\ell(Q)) \left\{ \frac{1}{|G(Q')|} \int_Q \left(\sum_{j=0}^{\infty} 2^{js} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda'_{jm'} \chi_{E(Q_{jm'}) \cap Q}(x) \right)^q dx \right\}^{\frac{1}{q}} \\ & \lesssim \|\lambda'\|_{\mathcal{M}_q^{s*,q}}, \end{aligned}$$

as was to be shown. \square

5.2. Pointwise multiplication. In this section we shall prove the following boundedness of pointwise multiplication operators.

Theorem 5.4 (Pointwise multiplication). *Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $\varphi \in \mathcal{G}_q$.*

- (1) *If $k > s > \sigma_r$, then the mapping*

$$g \in \mathcal{S}(\mathbb{R}^n) \mapsto g \cdot f \in \text{BC}^k(\mathbb{R}^n)$$

extends continuously to a bounded linear operator from $\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ to itself so that

$$\|g \cdot f\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|g\|_{\text{BC}^k} \|f\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s}$$

for all $f \in \mathcal{N}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ and $g \in \text{BC}^k(\mathbb{R}^n)$.

- (2) *If $k > s > \sigma_{qr}$, then the mapping*

$$g \in \mathcal{S}(\mathbb{R}^n) \mapsto g \cdot f \in \text{BC}^k(\mathbb{R}^n)$$

extends continuously to a bounded linear operator from $\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ to itself so that

$$\|g \cdot f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|g\|_{\text{BC}^k} \|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s}$$

for all $f \in \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ and $g \in \text{BC}^k(\mathbb{R}^n)$.

Proof. We shall concentrate on generalized Triebel-Lizorkin-Morrey spaces, since we can handle generalized Besov-Morrey spaces similarly. Let $\varepsilon \in (0, \rho - R)$. Let $f \in \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then there exists a triply indexed complex sequence $\lambda = \{\lambda_{\nu m}^\beta\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n}$ satisfying (4.32) and (4.33). Then we claim that

$$g \cdot f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta [g(\beta q u)_{\nu m}]$$

makes sense; the right-hand side is convergent in $\mathcal{S}'(\mathbb{R}^n)$ and satisfies the desired estimate. If we set

$$f^\beta \equiv \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta q u)_{\nu m}$$

for each $\beta \in \mathbb{N}_0^n$, then we have

$$\|f^\beta\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} \lesssim 2^{-(\rho-R-\varepsilon)|\beta|} \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s, \rho}.$$

By using the atomic decomposition theorem, we obtain

$$\|g \cdot f^\beta\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} \lesssim 2^{-(\rho-R-\varepsilon)|\beta|} \|g\|_{\text{BC}^k} \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s, \rho} \lesssim 2^{-(\rho-R-\varepsilon)|\beta|} \|g\|_{\text{BC}^k} \|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s}.$$

This estimate is summable over all $\beta \in \mathbb{N}_0^n$ with the desired estimate. \square

5.3. Diffeomorphism. A C^M -diffeomorphism $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be *regular*, if ψ and its inverse belong to $\text{BC}^M(\mathbb{R}^n)$.

Theorem 5.5 (Diffeomorphism). *Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $\varphi \in \mathcal{G}_q$. Assume in addition that ψ is a regular C^k -diffeomorphism.*

- (1) *Let $k > s > \sigma_q$. Then, the composition mapping $\varphi \in \text{BC}^k(\mathbb{R}^n) \mapsto \varphi \circ \psi \in \text{BC}^k(\mathbb{R}^n)$ induces a continuous mapping $f \in \mathcal{N}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \mapsto f \circ \psi \in \mathcal{N}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ and, for all $f \in \mathcal{N}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$, we have $\|f \circ \psi\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s} \lesssim_\psi \|f\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s}$.*
- (2) *Let $k > s > \sigma_{qr}$. Then, the composition mapping $\varphi \in \text{BC}^k(\mathbb{R}^n) \mapsto \varphi \circ \psi \in \text{BC}^k(\mathbb{R}^n)$ induces a continuous mapping $f \in \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \mapsto f \circ \psi \in \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ and, for all $f \in \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$, we have $\|f \circ \psi\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} \lesssim_\psi \|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s}$.*

For the proof of Theorem 5.5, we need a setup. Now that ψ is bi-Lipschitz, that is, both ψ and ψ^{-1} are Lipschitz continuous, there exist $I \in \mathbb{N}$ and $D > 0$ depending on ψ such that; for each ν , \mathbb{Z}^n is partitioned into $M_1^\nu, M_2^\nu, \dots, M_I^\nu$, and there exist injections

$$(5.18) \quad \iota_1^\nu : M_1^\nu \rightarrow \mathbb{Z}^n, \iota_2^\nu : M_2^\nu \rightarrow \mathbb{Z}^n, \dots, \iota_I^\nu : M_I^\nu \rightarrow \mathbb{Z}^n$$

such that; for all $i = 1, 2, \dots, I$, $\nu \in \mathbb{N}_0$ and multi-index $\beta \in \mathbb{N}_0^n$, we have

$$\psi^{-1}(\text{supp}((\beta qu)_{\nu m})) \subset D Q_{\nu \iota_i^\nu(m)}.$$

Note that ι_i^ν is a bijection from M_i^ν to $\iota_i^\nu(M_i^\nu)$. For $i = 1, 2, \dots, I$ and $\nu \in \mathbb{N}_0$, we write $\theta_i^\nu \equiv (\iota_i^\nu)^{-1}$.

The integer I is independent of ν as the following lemma shows:

Lemma 5.6. *We have a bound*

$$(5.19) \quad I \lesssim 1,$$

where the implicit constant does not depend on ν .

Proof. Since $\nabla \psi, \nabla[\psi^{-1}]$ are bounded functions,

$$(\|\nabla[\psi^{-1}]\|_\infty)^{-1}|x - y| \leq |\psi(x) - \psi(y)| \leq \|\nabla \psi\|_\infty |x - y|.$$

Here and below, we set

$$C_0 \equiv \max\{\|\nabla \psi\|_\infty, \|\nabla[\psi^{-1}]\|_\infty\}.$$

Then, $C_0^{-1}|x - y| \leq |\psi(x) - \psi(y)| \leq C_0|x - y|$. Fix $m_0 \in \mathbb{Z}^n$. Once we show that the number of $m \in \mathbb{Z}^n$ satisfying

$$\psi^{-1}(\text{supp}(\beta qu)_{\nu m_0}) \subset D Q_{\nu m}$$

is bounded, we obtain the estimate of I from above.

The diameter of $\psi^{-1}(\text{supp}(\beta qu)_{\nu m_0})$, which is given by

$$\sup\{|x - y| : x, y \in \psi^{-1}(\text{supp}(\beta qu)_{\nu m_0})\},$$

satisfies $2^{\nu+1}r \times C_0$. Let $|\text{supp}((\beta qu)_{\nu m})| \leq (2r)^n$. Then $\{D Q_{\nu m}\}_{m \in \mathbb{Z}^n}$ overlaps at most $[D + 2]^n$ times. That is,

$$\sum_{m \in \mathbb{Z}^n} \chi_{D Q_{\nu m}} \leq [D + 2]^n.$$

Hence,

$$\psi^{-1}(\text{supp}(\beta qu)_{\nu m_0})$$

intersects at most $[\sqrt{n} \times 2^1 r \times C_0 + 1]^n \times [D + 2]^n$ cubes belonging to $\{D Q_{\nu m}\}_{m \in \mathbb{Z}^n}$. Thus, we conclude $I \leq [\sqrt{n} \times 2^1 r \times C_0 + 1]^n \times [D + 2]^n$ and that the proof is complete. \square

Proof. We concentrate on the generalized Triebel-Lizorkin-Morrey space $\mathcal{E}_{\mathcal{M}_q^s, r}^s(\mathbb{R}^n)$. Then, maintaining the notation of Theorem 4.17, we let $\rho > R$. We shall invoke the quarkonial decomposition; see Theorem 4.17. We expand

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta qu)_{\nu m}.$$

Here the coefficient $\lambda = \{\lambda_{\nu m}^\beta\}_{\beta \in \mathbb{N}_0^n, (\nu, m) \in \mathbb{N}_0 \times \mathbb{Z}^n}$ satisfies

$$(5.20) \quad \|\lambda\|_{\mathbf{e}_{\mathcal{M}_q^s, r, \rho}^s} \lesssim \|f\|_{\mathcal{E}_{\mathcal{M}_q^s, r}^s}.$$

Here, we set

$$\lambda_{\nu, \bar{m}}^{\beta, i} \equiv \begin{cases} \lambda_{\nu, \theta_i^\nu(\bar{m})}^\beta & \bar{m} \in \iota_i^\nu(M_i^\nu), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad (\beta qu)_{\nu, \bar{m}}^i \equiv \begin{cases} (\beta qu)_{\nu, \theta_i^\nu(\bar{m})} \circ \psi & \bar{m} \in \iota_i^\nu(M_i^\nu), \\ 0 & \text{otherwise.} \end{cases}$$

Then, we want to define

$$(5.21) \quad f \circ \psi = \sum_{i=1}^I \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu \in \mathbb{N}_0} \sum_{\overline{m} \in \mathbb{Z}^n} \lambda_{\nu \overline{m}}^{\beta, i} (\beta q u)_{\nu \overline{m}}^i.$$

Let us verify that the infinite sum defining (5.21) makes sense. Set

$$f^{i, \beta} \equiv \sum_{\nu \in \mathbb{N}_0} \sum_{\overline{m} \in \mathbb{Z}^n} \lambda_{\nu \overline{m}}^{\beta, i} (\beta q u)_{\nu \overline{m}}^i$$

for $\beta \in \mathbb{N}_0^n$, $i = 1, 2, \dots, I$. Then, Theorem 4.5 yields

$$\|f^{i, \beta}\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} \lesssim_\psi 2^{(R+\varepsilon)|\beta|} \|\{\lambda_{\nu \overline{m}}^{\beta, i}\}_{\nu \in \mathbb{N}_0, \overline{m} \in \mathbb{Z}^n}\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s} \lesssim_\psi 2^{(R+\varepsilon)|\beta|} \|\lambda^\beta\|_{\mathbf{e}_{\mathcal{M}_q^\varphi, r}^s},$$

where $\lambda^\beta \equiv \{\lambda_{\nu \overline{m}}^\beta\}_{(\nu, \overline{m}) \in \mathbb{N}_0 \times \mathbb{Z}^n}$.

Let $\delta = \rho - R - \varepsilon > 0$. By the estimate of the quarkonial decompositions,

$$\|f^{i, \beta}\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} \lesssim_\psi 2^{(R+\varepsilon)|\beta|} \|f^\beta\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} \lesssim_\psi 2^{-\delta|\beta|} \|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s}.$$

Hence, if we use the $\min(q, r, 1)$ -triangle inequality to the sum

$$f \circ \psi = \sum_{i=1}^I \sum_{\beta \in \mathbb{N}_0^n} f^{i, \beta},$$

then we have $\|f \circ \psi\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s}$. Hence, (5.21) defines $f \circ \psi$. \square

6. HOMOGENEOUS SPACES

6.1. The space $\mathcal{S}'_\infty(\mathbb{R}^n)$. Our results in this paper carry over to the homogeneous setting.

Definition 6.1. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi : (0, \infty) \rightarrow (0, \infty)$ be a function in \mathcal{G}_q . Let τ be compactly supported functions satisfying

$$0 \notin \text{supp}(\tau), \quad \tau(\xi) > 0 \text{ if } \xi \in Q(2) \setminus Q(1).$$

define $\tau_k(\xi) \equiv \tau(2^{-k}\xi)$ for $\xi \in \mathbb{R}^n$ and $k \in \mathbb{Z}$.

- (1) The (homogeneous) generalized Besov-Morrey space $\dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{\dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s} \equiv \begin{cases} \left(\sum_{j=-\infty}^{\infty} 2^{jsr} \|\tau_j(D)f\|_{\mathcal{M}_q^\varphi}^r \right)^{\frac{1}{r}} & (r < \infty), \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\tau_j(D)f\|_{\mathcal{M}_q^\varphi} & (r = \infty) \end{cases}$$

is finite.

- (2) The (homogeneous) generalized Triebel-Lizorkin-Morrey space $\dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{\dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s} \equiv \begin{cases} \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jsr} |\tau_j(D)f|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi} & (r < \infty), \\ \left\| \sup_{j \in \mathbb{Z}} 2^{js} |\tau_j(D)f| \right\|_{\mathcal{M}_q^\varphi} & (r = \infty) \end{cases}$$

is finite.

(3) The space $\dot{\mathcal{A}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ denotes either $\dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ or $\dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$.

Theorem 6.2. Assume (2.4) in the case when $\dot{\mathcal{A}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) = \dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ with $r < \infty$. Then different choices of admissible τ will yield equivalent norms.

Proof. The proof is almost the same as Theorem 1.4. We indicate the necessary change. Define τ_k and $\tilde{\tau}_k$ by (3.1) and (3.2), respectively. Here, we let $k \in \mathbb{Z}$ instead of $k \in \mathbb{N}$. Then we can prove (3.5) by mimicking the proof of Theorem 1.4. Further details are omitted. \square

6.2. Atomic decomposition. We can consider atomic decompositions for the homogeneous spaces. The proof is similar to the nonhomogeneous case. So, we outline the proof based on the nonhomogeneous case.

Definition 6.3. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$.

(1) The (homogeneous) generalized Besov-Morrey sequence space $\dot{\mathbf{n}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ is the set of all doubly indexed complex sequences $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n}$ for which the quasi-norm

$$\|\lambda\|_{\dot{\mathbf{n}}_{\mathcal{M}_q^\varphi, r}^s} \equiv \begin{cases} \left(\sum_{j=-\infty}^{\infty} 2^{jsr} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{jm} \chi_{Q_{jm}} \right\|_{\mathcal{M}_q^\varphi}^r \right)^{\frac{1}{r}} & (r < \infty), \\ \sup_{j \in \mathbb{Z}} 2^{js} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{jm} \chi_{Q_{jm}} \right\|_{\mathcal{M}_q^\varphi} & (r = \infty) \end{cases}$$

is finite.

(2) The (homogeneous) generalized Triebel-Lizorkin-Morrey sequence space $\dot{\mathbf{e}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ is the set of all doubly indexed complex sequences $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n}$ for which the quasi-norm

$$\|\lambda\|_{\dot{\mathbf{e}}_{\mathcal{M}_q^\varphi, r}^s} \equiv \begin{cases} \left\| \left\{ \sum_{j=-\infty}^{\infty} 2^{jsr} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}| \chi_{Q_{jm}} \right)^r \right\}^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi} & (r < \infty), \\ \left\| \sup_{j \in \mathbb{Z}} 2^{js} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}| \chi_{Q_{jm}} \right) \right\|_{\mathcal{M}_q^\varphi} & (r = \infty) \end{cases}$$

is finite.

(3) The space $\dot{\mathcal{A}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ denotes either $\dot{\mathbf{n}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ or $\dot{\mathbf{e}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Assume (2.4) in the case when $\dot{\mathcal{A}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) = \dot{\mathbf{e}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ with $r < \infty$.

Definition 6.4. Let $L \in \mathbb{N}_0 \cup \{-1\}$ and $K \in \mathbb{N}_0$.

- (1) Let $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. A C^K -function $a : \mathbb{R}^n \rightarrow \mathbb{C}$ is said to be a (K, L) -atom supported near Q_{jm} , if (4.2) with $|\alpha| \leq K$ and (4.3) with $|\beta| \leq L$ hold. When $L = -1$, it is understood that (4.3) is a void condition.
- (2) Denote by $\mathfrak{A} = \mathfrak{A}(\mathbb{R}^n)$ the set of all collections $\{a_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n}$ of C^K -functions such that each a_{jm} is a (K, L) -atom supported near Q_{jm} .

Theorem 6.5. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$. Let also $L \in \mathbb{N}_0 \cup \{-1\}$ and $K \in \mathbb{N}_0$ satisfy

$$K \geq [1 + s]_+, \quad L \geq \max(-1, [\sigma_q - s]),$$

where $\sigma_q \equiv n \left(\frac{1}{q} - 1 \right)_+$.

- (1) Let $f \in \dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then there exist a family $\{a_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{\mathfrak{A}}$ and a doubly indexed complex sequence $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{\mathbf{n}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ such that

$$(6.1) \quad f = \sum_{j=-\infty}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right) \text{ in } \mathcal{S}'_{\infty}(\mathbb{R}^n)$$

and that

$$\|\lambda\|_{\dot{\mathbf{n}}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|f\|_{\dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s}.$$

- (2) Let $\{a_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{\mathfrak{A}}$ and $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{\mathbf{n}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then

$$f \equiv \sum_{j=-\infty}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right)$$

converges in $\mathcal{S}'_{\infty}(\mathbb{R}^n)$ and belongs to $\dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Furthermore,

$$(6.2) \quad \|f\|_{\dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|\lambda\|_{\dot{\mathbf{n}}_{\mathcal{M}_q^\varphi, r}^s}.$$

Proof. (1) is already obtained in [30, Theorem 10.15]. The proof of (2) is almost the same as Theorem 4.4. The only difference is the care of the convergence of the sum $\sum_{j=-J}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right)$ as $J \rightarrow \infty$.

By invoking (4.10), we can prove (6.2) when there exists J such that $\lambda_{jm} = 0$ for all $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$ with $|j| \geq J$. Let $\delta > 0$ be given by (4.11). Then we have

$$\left\| \sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right\|_{\dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^{s+\delta/2}} \lesssim 2^{j\delta/2} \|\lambda\|_{\dot{\mathbf{n}}_{\mathcal{M}_q^\varphi, r}^s}$$

for all $j \in \mathbb{Z} \cap (-\infty, 0]$ from Corollary 4.11 and (6.2). Thus, since we can show that $\dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^{s+\delta/2}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'_{\infty}(\mathbb{R}^n)$ analogously to the nonhomogeneous case,

$$f_- \equiv \sum_{j=-J}^0 \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right)$$

is convergent in $\mathcal{S}'_{\infty}(\mathbb{R}^n)$. Thus, we are in the position of applying the Fatou property as we did in (4.14). \square

Theorem 6.6. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi : (0, \infty) \rightarrow (0, \infty) \in \mathcal{G}_q$. Let also $L \in \mathbb{N}_0 \cup \{-1\}$ and $K \in \mathbb{N}_0$. Assume

$$K \geq [1+s]_+, \quad L \geq \max(-1, [\sigma_{qr} - s]),$$

where $\sigma_{qr} \equiv \max(\sigma_q, \sigma_r)$.

- (1) Let $f \in \dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then there exist a family $\{a_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{\mathfrak{A}}$ and a doubly indexed complex sequence $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{\mathbf{e}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ satisfying (6.1) and that

$$(6.3) \quad \|\lambda\|_{\dot{\mathbf{e}}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|f\|_{\dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s}.$$

- (2) Let $\{a_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{\mathfrak{A}}$ and $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{\mathbf{e}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then

$$f \equiv \sum_{j=-\infty}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right)$$

converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and belongs to $\dot{\mathcal{E}}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$. Furthermore,

$$\|f\|_{\dot{\mathcal{E}}^s_{\mathcal{M}^\varphi_q,r}} \lesssim \|\lambda\|_{\dot{\mathcal{E}}^s_{\mathcal{M}^\varphi_q,r}}.$$

Proof. Combine the ideas of Theorems 4.5 and 6.5. \square

6.3. Molecular decomposition. As a direct corollary of Theorems 6.8 and 6.9, we can show that $\mathcal{S}_\infty(\mathbb{R}^n) \subset \dot{\mathcal{A}}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$.

Definition 6.7. Let $L \in \mathbb{N}_0 \cup \{-1\}$ and $K, N \in \mathbb{N}_0$ be such that $N > K + n$.

- (1) Let $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. A C^K -function $b : \mathbb{R}^n \rightarrow \mathbb{C}$ is said to be a (K, L, N) -molecule supported near Q_{jm} , if (4.2) with $|\alpha| \leq K$ and (4.3) with $|\beta| \leq L$ hold. When $L = -1$, it is understood that (4.3) is a void condition.
- (2) Denote by $\mathfrak{M} = \mathfrak{M}(\mathbb{R}^n)$ the set of all collections $\{b_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n}$ of C^K -functions such that each b_{jm} is a (K, L, N) -molecule supported near Q_{jm} .

Theorem 6.8. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$. Let also $L \in \mathbb{N}_0 \cup \{-1\}$ and $K \in \mathbb{N}_0$. Assume

$$K \geq [1 + s]_+, \quad L \geq \max(-1, [\sigma_q - s]),$$

where $\sigma_q \equiv n \left(\frac{1}{q} - 1 \right)_+$.

- (1) Let $f \in \dot{\mathcal{N}}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$. Then there exist a family $\{b_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \mathfrak{M}$ and a doubly indexed complex sequence $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{\mathbf{n}}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$ such that

$$(6.4) \quad f = \sum_{j=-\infty}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} b_{jm} \right) \text{ in } \mathcal{S}'_\infty(\mathbb{R}^n)$$

and that

$$\|\lambda\|_{\dot{\mathbf{n}}^s_{\mathcal{M}^\varphi_q,r}} \lesssim \|f\|_{\dot{\mathcal{N}}^s_{\mathcal{M}^\varphi_q,r}}.$$

- (2) Let $\{b_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \mathfrak{M}$ and $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{\mathbf{n}}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$. Then

$$f \equiv \sum_{j=-\infty}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} b_{jm} \right)$$

converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and belongs to $\dot{\mathcal{N}}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$. Furthermore,

$$\|f\|_{\dot{\mathcal{N}}^s_{\mathcal{M}^\varphi_q,r}} \lesssim \|\lambda\|_{\dot{\mathbf{n}}^s_{\mathcal{M}^\varphi_q,r}}.$$

Proof. Combine the ideas of Theorems 4.9 and 6.5. \square

Theorem 6.9. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$. Let also $K \in \mathbb{N}_0$ and $L \in \mathbb{N}_0 \cup \{-1\}$. Assume

$$K \geq [1 + s]_+, \quad L \geq \max(-1, [\sigma_{qr} - s]),$$

where $\sigma_{qr} \equiv \max(\sigma_q, \sigma_r)$.

- (1) Let $f \in \dot{\mathcal{E}}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$. Then there exist a family $\{b_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \mathfrak{M}$ and a doubly indexed complex sequence $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{\mathcal{E}}^s_{\mathcal{M}^\varphi_q,r}(\mathbb{R}^n)$ satisfying (6.4) and that

$$(6.5) \quad \|\lambda\|_{\dot{\mathcal{E}}^s_{\mathcal{M}^\varphi_q,r}} \lesssim \|f\|_{\dot{\mathcal{E}}^s_{\mathcal{M}^\varphi_q,r}}.$$

(2) Let $\{b_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{\mathfrak{M}}$ and $\lambda = \{\lambda_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then

$$f \equiv \sum_{j=-\infty}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} b_{jm} \right)$$

converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and belongs to $\dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Furthermore,

$$\|f\|_{\dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|\lambda\|_{\dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s}.$$

Proof. Combine the ideas of Theorems 4.10 and 6.5. \square

6.4. Quarkonial decomposition. As we have seen in the nonhomogeneous case, quarkonial decomposition can be obtained on the basis of the atomic decomposition. We content ourselves with indicating how to modify the related definition and stating our results without proofs.

Definition 6.10 (Regular quark). Let $\beta \in \mathbb{N}_0^n$, $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then define a function ψ^β and the quark $(\beta qu)_{\nu m} \equiv (\beta qu)_{\nu, m}$ by (4.25). Each $(\beta qu)_{\nu m}$ is called the *quark*.

Definition 6.11 (Sequence spaces for quarkonial decomposition). Let $R, \rho > 0$ satisfy (4.24) and $\rho > R$. For a triply indexed complex sequence $\lambda = \{\lambda_{\nu m}^\beta\}_{\beta \in \mathbb{N}_0^n, \nu \in \mathbb{Z}, m \in \mathbb{Z}^n}$, define

$$(6.6) \quad \lambda^\beta \equiv \{\lambda_{\nu m}^\beta\}_{\nu \in \mathbb{Z}, m \in \mathbb{Z}^n}, \quad \|\lambda\|_{\dot{\mathfrak{M}}_{\mathcal{M}_q^\varphi, r}^s, \rho} \equiv \sup_{\beta \in \mathbb{N}_0^n} 2^{\rho|\beta|} \|\lambda^\beta\|_{\dot{\mathfrak{M}}_{\mathcal{M}_q^\varphi, r}^s}.$$

Theorem 6.12. Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$ and $\varphi \in \mathcal{G}_q$. Assume $\rho \equiv [R+1] > R$, where R is a constant in (4.24).

(1) Let $s > \sigma_q$ and $f \in \dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Then there exists a triply indexed complex sequence

$$\lambda = \{\lambda_{\nu m}^\beta\}_{\nu \in \mathbb{Z}, m \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n}$$

such that

$$(6.7) \quad f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta qu)_{\nu m}$$

converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and

$$(6.8) \quad \|\lambda\|_{\dot{\mathfrak{M}}_{\mathcal{M}_q^\varphi, r}^s, \rho} \lesssim \|f\|_{\dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s}.$$

The constant $\lambda_{\nu m}^\beta$ depends continuously and linearly on f .

(2) If $s > \sigma_q$ and $\lambda = \{\lambda_{\nu m}^\beta\}_{\nu \in \mathbb{Z}, m \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n}$ satisfies $\|\lambda\|_{\dot{\mathfrak{M}}_{\mathcal{M}_q^\varphi, r}^s, \rho} < \infty$, then

$$f \equiv \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta qu)_{\nu m}$$

converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and belongs to $\dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$. Furthermore,

$$\|f\|_{\dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s} \lesssim \|\lambda\|_{\dot{\mathfrak{M}}_{\mathcal{M}_q^\varphi, r}^s, \rho}.$$

(3) If $s > \sigma_{qr}$ and $f \in \dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$, then there exists a triply indexed complex sequence $\lambda = \{\lambda_{\nu m}^\beta\}_{\nu \in \mathbb{Z}, m \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n}$ such that

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta qu)_{\nu m}$$

in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and

$$\|\lambda\|_{\dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s, \rho} \lesssim \|f\|_{\dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s}.$$

The constant $\lambda_{\nu m}^\beta$ depends continuously and linearly on f .

(4) If $s > \sigma_{qr}$ and $\lambda = \{\lambda_{\nu m}^\beta\}_{\nu \in \mathbb{Z}, m \in \mathbb{Z}^n, \beta \in \mathbb{N}_0^n}$ satisfies $\|\lambda\|_{\dot{\mathcal{M}}_{q,r,\rho}^s} < \infty$, then

$$f \equiv \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta q u)_{\nu m}$$

converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and belongs to $\dot{\mathcal{E}}_{\mathcal{M}_{q,r}^s}^s(\mathbb{R}^n)$. Furthermore,

$$\|f\|_{\dot{\mathcal{E}}_{\mathcal{M}_{q,r}^s}^s} \lesssim \|\lambda\|_{\dot{\mathcal{M}}_{q,r,\rho}^s}.$$

Proof. Let us prove (1) and (3). We modify (4.37) by using (4.36) as follows:

$$f = \frac{1}{\sqrt{(2\pi)^n}} \sum_{\nu=-\infty}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \varphi_\nu(D) f(2^{-\nu} m) \mathcal{F}^{-1} \kappa(2^\nu \cdot -m) \right),$$

where the convergence takes place in $\mathcal{S}'_\infty(\mathbb{R}^n)$. Then, we go through the same argument as Theorem 4.17.

We can prove (2) and (4) by using Theorems 6.5 and 6.6 instead of Theorems 4.4 and 4.5. \square

6.5. Fundamental theorems—trace operator, pointwise multiplication and diffeomorphism. The results are obtained similarly to the ones for nonhomogeneous spaces; we content ourselves with stating the results.

Theorem 6.13. *Let $n \geq 2$. Suppose that we are given parameters $(q, r, s) \in (0, \infty) \times (0, \infty] \times (0, \infty)$ and a function $\varphi \in \mathcal{G}_q$. Define s^* and φ^* by (5.2) and (5.3)*

- (1) *Let s satisfy (5.5). Then we can extend the trace operator $f \in \mathcal{S}_\infty(\mathbb{R}^n) \mapsto f(\cdot, 0_n) \in C^\infty(\mathbb{R}^n)$ to a bounded surjective linear operator from $\dot{\mathcal{N}}_{\mathcal{M}_{q,r}^\varphi}^s(\mathbb{R}^n)$ to $\dot{\mathcal{N}}_{\mathcal{M}_{q,r}^{\varphi^*}}^{s^*}(\mathbb{R}^{n-1})$.*
- (2) *Let s satisfy (5.6). Then we can extend the trace operator $f \in \mathcal{S}_\infty(\mathbb{R}^n) \mapsto f(\cdot, 0_n) \in C^\infty(\mathbb{R}^n)$ to a bounded surjective linear operator from $\dot{\mathcal{E}}_{\mathcal{M}_{q,r}^\varphi}^s(\mathbb{R}^n)$ to $\dot{\mathcal{E}}_{\mathcal{M}_{q,r}^{\varphi^*}}^{s^*}(\mathbb{R}^{n-1})$.*

Proof. We just mimic the proof of Theorems 5.1 and 5.3 by replacing \mathbb{N}_0 by \mathbb{Z} and $j'_Q = \max(0, -\log_2 \ell(Q))$ by $j_Q \equiv -\log_2 \ell(Q)$, respectively. \square

Theorem 6.14 (Pointwise multiplication). *Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $\varphi \in \mathcal{G}_q$.*

- (1) *If $k > s > \sigma_r$, then the mapping*

$$g \in \mathcal{S}_\infty(\mathbb{R}^n) \mapsto g \cdot f \in \text{BC}^k(\mathbb{R}^n)$$

extends continuously to a bounded linear operator from $\dot{\mathcal{N}}_{\mathcal{M}_{q,r}^\varphi}^s(\mathbb{R}^n)$ to itself so that

$$\|g \cdot f\|_{\dot{\mathcal{N}}_{\mathcal{M}_{q,r}^\varphi}^s} \lesssim \|g\|_{\text{BC}^k} \|f\|_{\dot{\mathcal{N}}_{\mathcal{M}_{q,r}^\varphi}^s}$$

for all $f \in \dot{\mathcal{N}}_{\mathcal{M}_{q,r}^\varphi}^s(\mathbb{R}^n)$ and $g \in \text{BC}^k(\mathbb{R}^n)$.

- (2) *If $k > s > \sigma_{qr}$, then the mapping*

$$g \in \mathcal{S}_\infty(\mathbb{R}^n) \mapsto g \cdot f \in \text{BC}^k(\mathbb{R}^n)$$

extends continuously to a bounded linear operator from $\dot{\mathcal{E}}_{\mathcal{M}_{q,r}^\varphi}^s(\mathbb{R}^n)$ to itself so that

$$\|g \cdot f\|_{\dot{\mathcal{E}}_{\mathcal{M}_{q,r}^\varphi}^s} \lesssim \|g\|_{\text{BC}^k} \|f\|_{\dot{\mathcal{E}}_{\mathcal{M}_{q,r}^\varphi}^s}$$

for all $f \in \dot{\mathcal{E}}_{\mathcal{M}_{q,r}^\varphi}^s(\mathbb{R}^n)$ and $g \in \text{BC}^k(\mathbb{R}^n)$.

Proof. Mimic the proof of Theorem 5.4. \square

Theorem 6.15 (Diffeomorphism). *Let $0 < q < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $\varphi \in \mathcal{G}_q$. Assume in addition that ψ is a regular C^k -diffeomorphism.*

- (1) *Let $k > s > \sigma_q$. Then, the composition mapping $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n) \mapsto \varphi \circ \psi \in \text{BC}^k(\mathbb{R}^n)$ induces a continuous mapping $f \in \dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \mapsto f \circ \psi \in \dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ and, for all $f \in \dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$, we have $\|f \circ \psi\|_{\dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s} \lesssim_\psi \|f\|_{\dot{\mathcal{N}}_{\mathcal{M}_q^\varphi, r}^s}$.*
- (2) *Let $k > s > \sigma_{qr}$. Then, the composition mapping $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n) \mapsto \varphi \circ \psi \in \text{BC}^k(\mathbb{R}^n)$ induces a continuous mapping $f \in \dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n) \mapsto f \circ \psi \in \dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$ and, for all $f \in \dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$, we have $\|f \circ \psi\|_{\dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s} \lesssim_\psi \|f\|_{\dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, r}^s}$.*

Proof. Mimic the proof of Theorem 5.5. \square

6.6. Generalized Hardy spaces and generalized Triebel-Lizorkin-Morrey spaces. Let $0 < q < \infty$ and $\varphi \in \mathcal{G}_q$. The generalized Hardy-Morrey space $H\mathcal{M}_q^\varphi(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying $\sup_{t>0} |e^{t\Delta} f(\cdot)| \in \mathcal{M}_q^\varphi(\mathbb{R}^n)$. We equip $H\mathcal{M}_q^\varphi(\mathbb{R}^n)$ with the following norm:

$$(6.9) \quad \|f\|_{H\mathcal{M}_q^\varphi} \equiv \left\| \sup_{t>0} |e^{t\Delta} f| \right\|_{\mathcal{M}_q^\varphi} \quad (f \in H\mathcal{M}_q^\varphi(\mathbb{R}^n)).$$

We invoke the following decomposition result from [1, Theorem 15] when $0 < q \leq 1$ and from [14, Theorem 1.1] when $1 \leq q < \infty$. Here \mathcal{Q} denotes the set of all cubes.

Lemma 6.16. *Let $0 < q \leq 1$, $\varphi \in \mathcal{G}_q$ and $f \in H\mathcal{M}_q^\varphi(\mathbb{R}^n)$. Let $L \in (\mathbb{N} \cup \{0\}) \cap [\sigma_q, \infty)$. Assume that $\varphi, \eta \in \mathcal{G}_1$ satisfy*

$$(6.10) \quad \int_r^\infty \varphi(s) \frac{ds}{s} \lesssim \varphi(r)$$

for $r > 0$. Then there exists a triplet $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$ and $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and that, for all $v > 0$

$$(6.11) \quad |a_j| \leq \frac{\chi_{Q_j}}{\|\chi_{Q_j}\|_{\mathcal{M}_q^\varphi}}, \quad \int_{\mathbb{R}^n} x^\alpha a_j(x) dx = 0, \quad \left\| \left(\sum_{j=1}^\infty \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_{\mathcal{M}_q^\varphi}} \chi_{Q_j} \right)^v \right)^{1/v} \right\|_{\mathcal{M}_q^\varphi} \lesssim_v \|f\|_{H\mathcal{M}_q^\varphi}$$

for all multi-indexes α with $|\alpha| \leq L$.

Going through the same argument as [42, Theorem 5.5] and [43, Theorem 5.5], we can prove the following theorem;

Lemma 6.17. *Let $0 < q < \infty$. Let $\varphi \in \mathcal{G}_q$ satisfy (6.10). Let $k \in \mathcal{S}(\mathbb{R}^n)$. Write*

$$A_m \equiv \sup_{x \in \mathbb{R}^n} |x|^{n+m} |\nabla^m k(x)| \quad (m \in \mathbb{N} \cup \{0\}).$$

Define a convolution operator T by

$$Tf(x) \equiv k * f(x) \quad (f \in \mathcal{S}'(\mathbb{R}^n)).$$

Then, T , restricted to $H\mathcal{M}_q^\varphi(\mathbb{R}^n)$, is an $H\mathcal{M}_q^\varphi(\mathbb{R}^n)$ -bounded operator and the norm depends only on $\|\mathcal{F}k\|_{L^\infty}$ and a finite number of collections A_1, A_2, \dots, A_N with N depending only on φ .

Proof. We follow [42, Proposition 5.3]. Let Q be a cube and b be a measurable function satisfying $|b| \leq \chi_Q$ and $\int_{\mathbb{R}^n} x^\alpha b(x) dx = 0$ for all $|\alpha| \leq L$. In view of the moment condition, we have

$$|e^{t\Delta}[k * b](x)| \lesssim \chi_{2Q}(x) |e^{t\Delta}[k * b](x)| + \frac{\ell(Q)^{n+L+1}}{\ell(Q)^{n+L+1} + |x - c(Q)|^{n+L+1}}$$

for all $t > 0$ and hence

$$\mathcal{M}[k * b](x) \lesssim \chi_{2Q}(x) \mathcal{M}[k * b](x) + \frac{\ell(Q)^{n+L+1}}{\ell(Q)^{n+L+1} + |x - c(Q)|^{n+L+1}},$$

as was to be shown. \square

Once Lemma 6.17 is proved, we can obtain the Littlewood-Paley decomposition in the same way as [42, Theorem 5.7] and [43, Theorem 5.10]. See [1, Theorem 3.8] when $0 < q \leq 1$. The same proof works when $1 < q < \infty$ since we have $H\mathcal{M}_q^\varphi(\mathbb{R}^n) = \mathcal{M}_q^\varphi(\mathbb{R}^n)$ from [14, Proposition 5.1].

Corollary 6.18. *Let $\varphi \in \mathcal{G}_q$ satisfy (6.10).*

- (1) *Let $0 < q \leq 1$. Then $\dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, 2}^0(\mathbb{R}^n) \simeq H\mathcal{M}_q^\varphi(\mathbb{R}^n)$.*
- (2) *Let $1 < q < \infty$. Then $\dot{\mathcal{E}}_{\mathcal{M}_q^\varphi, 2}^0(\mathbb{R}^n) \simeq \mathcal{M}_q^\varphi(\mathbb{R}^n)$.*

6.7. Observations of the space $\mathcal{S}'_\infty(\mathbb{R}^n)$. Since $\mathcal{S}_\infty(\mathbb{R}^n)$ is continuously embedded into $\mathcal{S}(\mathbb{R}^n)$, the dual operator R , called the restriction, is continuous from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'_\infty(\mathbb{R}^n)$. The following theorem is a folklore fact:

Theorem 6.19. *The restriction mapping $R : f \in \mathcal{S}'(\mathbb{R}^n) \mapsto f|_{\mathcal{S}'_\infty(\mathbb{R}^n)} \in \mathcal{S}'(\mathbb{R}^n)$ is open, namely the image $R(U)$ is open in $\mathcal{S}'_\infty(\mathbb{R}^n)$ for any open set U in $\mathcal{S}'(\mathbb{R}^n)$.*

However, the proof can not be found in any literature. It seems that we can not readily use the closed graph theorem for a certain class of topological vector spaces. We therefore supply the self-contained and elementary proof. Theorem 6.19 can be used to consider the function spaces of homogeneous type; see Section 7.

First, we will invoke the following statement of the Hahn-Banach theorem from [34, p. 75 1.9.7 Corollary].

Lemma 6.20. *Let Y be a closed subspace of a normed space X . Suppose that $x \in X \setminus Y$. Then, there is a bounded linear functional f on X such that $\|f\| = 1$, that $f(x) = d(x, Y)$ and that $Y \subset \ker(f)$.*

Denote by $\mathcal{V}_N(\mathbb{R}^n)$ the closure of $\mathcal{S}(\mathbb{R}^n)$ and by $\mathcal{V}_{N,\infty}(\mathbb{R}^n)$ the closure of $\mathcal{S}_\infty(\mathbb{R}^n)$, where the closure is considered with respect to p_N .

Remark 6.21. Since $\int_{\mathbb{R}^n} f(x) dx = 0$ for all $f \in \mathcal{V}_{N+1,\infty}(\mathbb{R}^n)$, the Gaussian function $f(x) = e^{-|x|^2}$ does not belong to $\mathcal{V}_{N+1,\infty}(\mathbb{R}^n)$. Thus, $\mathcal{S}(\mathbb{R}^n) \setminus \mathcal{V}_{N+1,\infty}(\mathbb{R}^n) \neq \emptyset$.

Proposition 6.22. *Let $\Phi_1, \Phi_2, \dots, \Phi_L \in \mathcal{V}_N(\mathbb{R}^n)$ be a finite sequence. Then the space $\mathcal{V}_{N,\infty}(\mathbb{R}^n) + \text{Span}(\{\Phi_1, \Phi_2, \dots, \Phi_L\})$ is a closed subspace of $\mathcal{V}_N(\mathbb{R}^n)$.*

Proof. We start with a setup. We may assume that the system

$$(6.12) \quad \mathcal{K} = \{[\Phi_1], [\Phi_2], \dots, [\Phi_L]\}$$

are linearly independent in $\mathcal{V}_N(\mathbb{R}^n)/\mathcal{V}_{N,\infty}(\mathbb{R}^n)$. According to Lemma 6.20, we can find a bounded linear functional $\ell_1 : \mathcal{V}_N(\mathbb{R}^n) \rightarrow \mathbb{C}$ such that $\mathcal{V}_{N,\infty}(\mathbb{R}^n) \subset \ker(\ell_1)$ and that $\ell_1(\Phi_1) = 1$. Inductively, we can construct $\ell_2, \dots, \ell_L : \mathcal{V}_N(\mathbb{R}^n) \rightarrow \mathbb{C}$ such that $\ell_l(\Phi_l) = 1$ and that $\mathcal{V}_{N,\infty}(\mathbb{R}^n) \cup \{\Phi_1, \Phi_2, \dots, \Phi_{l-1}\} \subset \ker(\ell_l)$. If we consider some linear combinations, we can suppose that $\ell_l(\Phi_{l'}) = \delta_{l,l'}$ for all $1 \leq l, l' \leq L$.

Let $\{\tau_k\}_{k=1}^\infty$ be a sequence in $\mathcal{V}_{N,\infty}(\mathbb{R}^n) + \text{Span}(\{\Phi_1, \Phi_2, \dots, \Phi_L\})$ convergent to $\tau \in \mathcal{V}_N(\mathbb{R}^n)$. Then, we have

$$\tau_k = \sum_{l=1}^N a_{lk} \Phi_l + \zeta_k$$

for some $\zeta_k \in \mathcal{V}_{N,\infty}(\mathbb{R}^n)$ and $a_{lk} \in \mathbb{C}, l = 1, 2, \dots, L$. In terms of the ℓ_l 's, we have

$$\tau_k = \sum_{l=1}^N \ell_l(\tau_k) \Phi_l + \zeta_k \text{ or equivalently } \zeta_k = \tau_k - \sum_{l=1}^N \ell_l(\tau_k) \Phi_l.$$

Since by letting $k \rightarrow \infty$, we have that $\zeta_k, k = 1, 2, \dots$ converges to a function $\zeta \in \mathcal{V}_N(\mathbb{R}^n)$. Since $\zeta_k \in \mathcal{V}_{N,\infty}(\mathbb{R}^n)$, we have $\zeta \in \mathcal{V}_{N,\infty}(\mathbb{R}^n)$. Thus,

$$\tau = \sum_{l=1}^N \ell_l(\tau) \Phi_l + \zeta \in \mathcal{V}_{N,\infty}(\mathbb{R}^n) + \text{Span}(\{\Phi_1, \Phi_2, \dots, \Phi_L\}),$$

as was to be shown. \square

The following lemma is somehow well known. However, for convenience for readers we supply the proof.

Lemma 6.23. *Let $g \in \mathcal{S}'_\infty(\mathbb{R}^n)$. Then there exists $N \in \mathbb{N}$ such that*

$$(6.13) \quad |\langle g, \varphi \rangle| \leq N p_N(\varphi)$$

for all $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$.

Proof. Suppose that $g : \mathcal{S}_\infty(\mathbb{R}^n) \rightarrow \mathbb{C}$ is continuous; our task is to find $N \in \mathbb{N}$ such that (6.13) holds. By the continuity of g , the set

$$(6.14) \quad g^{-1}(\{z \in \mathbb{C} : |z| < 1\}) = \{\varphi \in \mathcal{S}_\infty(\mathbb{R}^n) : |g(\varphi)| < 1\}$$

is an open set of $\mathcal{S}_\infty(\mathbb{R}^n)$ that contains 0.

Therefore, if $L \in \mathbb{N}$ is sufficiently large, we conclude

$$(6.15) \quad \{\varphi \in \mathcal{S}_\infty(\mathbb{R}^n) : L p_L(\varphi) < 1\} \subset \{\varphi \in \mathcal{S}_\infty(\mathbb{R}^n) : |g(\varphi)| < 1\}.$$

Hence, $|g(\varphi)| \leq 1$ as long as $\varphi \in \mathcal{S}$ satisfies $L p_{2L}(\varphi) = \frac{1}{2}$.

Now, we suppose that we are given $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n) \setminus \{0\}$. Then, $\psi \equiv \frac{1}{2L p_{2L}(\varphi)} \varphi$ satisfies $L p_{2L}(\psi) = \frac{1}{2}$. Thus, $|g(\psi)| \leq 1$. In view of the definition of ψ , we have $|g(\varphi)| \leq 2L p_{2L}(\varphi), \varphi \in \mathcal{S}_\infty(\mathbb{R}^n) \setminus \{0\}$. The case when $\varphi = 0$ can be readily incorporated. Thus, by letting $N = 2L$, we can choose N satisfying (6.13). \square

Note that $\mathcal{V}_N(\mathbb{R}^n)$ is a subset of all $C^N(\mathbb{R}^n)$ -functions and that

$$(6.16) \quad \int_{\mathbb{R}^n} x^\alpha g(x) dx = 0$$

for all $g \in \mathcal{V}_{N,\infty}(\mathbb{R}^n)$ and $|\alpha| \leq N - n - 1$. With this in mind, let us prove the following theorem:

Theorem 6.24. *Let $\varphi_1, \varphi_2, \dots, \varphi_K \in \mathcal{S}_\infty(\mathbb{R}^n)$ and $\Phi_1, \Phi_2, \dots, \Phi_L \in \mathcal{S}(\mathbb{R}^n) \setminus \mathcal{S}_\infty(\mathbb{R}^n)$. Assume in addition that $[\Phi_1], [\Phi_2], \dots, [\Phi_L] \in \mathcal{S}(\mathbb{R}^n) \setminus \mathcal{S}_\infty(\mathbb{R}^n)$ is linearly independent in $\mathcal{S}(\mathbb{R}^n)/\mathcal{S}_\infty(\mathbb{R}^n)$. Then the image of*

$$\mathcal{U} \equiv \left(\bigcap_{k=1}^K \{F \in \mathcal{S}'(\mathbb{R}^n) : |\langle F, \varphi_k \rangle| < 1\} \right) \cap \left(\bigcap_{l=1}^L \{F \in \mathcal{S}'(\mathbb{R}^n) : |\langle F, \Phi_l \rangle| < 1\} \right)$$

by R is exactly

$$U \equiv \bigcap_{k=1}^K \{f \in \mathcal{S}'_\infty(\mathbb{R}^n) : |\langle f, \varphi_k \rangle| < 1\}.$$

Proof. It is trivial that $R(\mathcal{U}) \subset U$. Let $f \in U$ to prove $R(\mathcal{U}) \supset U$. Since

$$\Phi_1, \Phi_2, \dots, \Phi_L \in \mathcal{S}(\mathbb{R}^n) \setminus \mathcal{S}_\infty(\mathbb{R}^n),$$

there exist $\alpha_1, \alpha_2, \dots, \alpha_L \in \mathbb{N}_0^n$ such that

$$(6.17) \quad \int_{\mathbb{R}^n} x^{\alpha_l} \Phi_l(x) dx \neq 0$$

Since $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, there exists an integer N such that

$$|\langle f, \varphi \rangle| \leq N p_N(\varphi)$$

for all $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$. Note that f continuously extends to $\mathcal{V}_{N,\infty}(\mathbb{R}^n)$. So, we regard f as a continuous linear mapping defined on $\mathcal{V}_{N,\infty}(\mathbb{R}^n)$.

If necessary by replacing N with a large one, we may assume

$$N > |\alpha_1| + |\alpha_2| + \dots + |\alpha_L| + n + 1.$$

It follows from (6.16) and (6.17) that

$$\Phi_l \in \mathcal{V}_N(\mathbb{R}^n) \setminus \mathcal{V}_{N,\infty}(\mathbb{R}^n) \quad (l = 1, 2, \dots, L).$$

Let p be the projection from $\mathcal{V}_{N,\infty}(\mathbb{R}^n) + \text{Span}(\{\Phi_1, \Phi_2, \dots, \Phi_L\})$ to $\mathcal{V}_{N,\infty}(\mathbb{R}^n)$. Let us set

$$H \equiv f \circ p : \mathcal{V}_{N,\infty}(\mathbb{R}^n) + \text{Span}(\{\Phi_1, \Phi_2, \dots, \Phi_L\}) \rightarrow \mathbb{C}.$$

Then since p is continuous and $\mathcal{V}_{N,\infty}(\mathbb{R}^n) + \text{Span}(\{\Phi_1, \Phi_2, \dots, \Phi_L\})$ is a closed subspace of $\mathcal{V}_N(\mathbb{R}^n)$, H extends to a bounded linear functional F on $\mathcal{V}_N(\mathbb{R}^n)$. Note that

$$(6.18) \quad \langle F, \Phi_l \rangle = \langle H, \Phi_l \rangle = 0$$

for all $l = 1, 2, \dots, L$ and that $f = R(F)$, which implies $F \in \mathcal{U}$ as well. Therefore, $R(\mathcal{U}) = U$. \square

Theorem 6.25. *Let $\varphi_1, \varphi_2, \dots, \varphi_K \in \mathcal{S}_\infty(\mathbb{R}^n)$ and $\Phi_1, \Phi_2, \dots, \Phi_L, \dots, \Phi_{L^*} \in \mathcal{S}(\mathbb{R}^n) \setminus \mathcal{S}_\infty(\mathbb{R}^n)$. Assume in addition that $[\Phi_1], [\Phi_2], \dots, [\Phi_L] \in \mathcal{S}(\mathbb{R}^n) \setminus \mathcal{S}_\infty(\mathbb{R}^n)$ is linearly independent in $\mathcal{S}(\mathbb{R}^n)/\mathcal{S}_\infty(\mathbb{R}^n)$ and that $\mathcal{S}_\infty(\mathbb{R}^n)$ and $\Phi_1, \Phi_2, \dots, \Phi_L \in \mathcal{S}(\mathbb{R}^n) \setminus \mathcal{S}_\infty(\mathbb{R}^n)$ span $\Phi_{L+1}, \dots, \Phi_{L^*}$. More precisely, we assume*

$$(6.19) \quad \Phi_l = \varphi_l^* + \sum_{k=1}^L \beta_{l,k} \Phi_k \quad (l = L+1, \dots, L^*)$$

for some $\varphi_{L+1}^*, \dots, \varphi_{L^*}^* \in \mathcal{S}_\infty(\mathbb{R}^n)$. Then the image of

$$\mathcal{U} \equiv \left(\bigcap_{k=1}^K \{F \in \mathcal{S}'(\mathbb{R}^n) : |\langle F, \varphi_k \rangle| < 1\} \right) \cap \left(\bigcap_{l=1}^{L^*} \{F \in \mathcal{S}'(\mathbb{R}^n) : |\langle F, \Phi_l \rangle| < 1\} \right)$$

by R contains

$$U \equiv \left(\bigcap_{k=1}^K \{f \in \mathcal{S}'_\infty(\mathbb{R}^n) : |\langle f, \varphi_k \rangle| < 1\} \right) \cap \left(\bigcap_{k=L+1}^{L^*} \{f \in \mathcal{S}'_\infty(\mathbb{R}^n) : |\langle f, \varphi_k^* \rangle| < 1\} \right).$$

Proof. We know that the image of

$$\begin{aligned} \tilde{\mathcal{U}} \equiv & \left(\bigcap_{k=1}^K \{F \in \mathcal{S}'(\mathbb{R}^n) : |\langle F, \varphi_k \rangle| < 1\} \right) \cap \left(\bigcap_{l=1}^L \{F \in \mathcal{S}'(\mathbb{R}^n) : |\langle F, \Phi_l \rangle| < 1\} \right) \\ & \cap \left(\bigcap_{k=L+1}^{L^*} \{F \in \mathcal{S}'(\mathbb{R}^n) : |\langle F, \varphi_k^* \rangle| < 1\} \right) \end{aligned}$$

by R is exactly

$$U \equiv \left(\bigcap_{k=1}^K \{f \in \mathcal{S}'_\infty(\mathbb{R}^n) : |\langle f, \varphi_k \rangle| < 1\} \right) \cap \left(\bigcap_{k=L+1}^{L^*} \{f \in \mathcal{S}'_\infty(\mathbb{R}^n) : |\langle f, \varphi_k^* \rangle| < 1\} \right)$$

thanks to Theorem 6.24. According to (6.18) and (6.19), we can say that the image of

$$\begin{aligned} \tilde{\mathcal{U}}^* \equiv & \left(\bigcap_{k=1}^K \{F \in \mathcal{S}'(\mathbb{R}^n) : |\langle F, \varphi_k \rangle| < 1\} \right) \cap \left(\bigcap_{l=1}^{L^*} \{F \in \mathcal{S}'(\mathbb{R}^n) : |\langle F, \Phi_l \rangle| < 1\} \right) \\ & \cap \left(\bigcap_{k=L+1}^{L^*} \{F \in \mathcal{S}'(\mathbb{R}^n) : |\langle F, \varphi_k^* \rangle| < 1\} \right) \end{aligned}$$

is U . Since \mathcal{U} contains $\tilde{\mathcal{U}}^*$, the image of \mathcal{U} by R contains U . \square

Now the proof of Theorem 6.19 is easy. In fact, let \mathcal{U}_0 be a neighborhood of 0. Then according to Theorem 6.25, \mathcal{U}_0 contains a set of the form \mathcal{U} described in Theorem 6.25. According to Theorem 6.25, we know that $0 \in U = R(\mathcal{U}) \subset R(\mathcal{U}_0)$. Thus, 0 is an interior point of $R(\mathcal{U}_0)$. By the translation, we can show that any point $R(f)$ with $f \in R(\mathcal{U}_0)$ can be proved to be an interior point of $R(\mathcal{U}_0)$.

Let us rephrase Theorem 6.19 in terms of the quotient topology. To begin with let us recall some elementary facts on general topology.

Definition 6.26. An equivalence relation of a set X is a subset R of $X \times X$ satisfying the following. Below, for $x, y \in X$, $x \sim y$ means that $(x, y) \in R$.

- (1) $x \sim x$ for all $x \in X$ (Reflexivity).
- (2) Let $x, y \in X$. Then $x \sim y$ implies $y \sim x$ (Symmetry).
- (3) Let $x, y, z \in X$. Then $x \sim y$ and $y \sim z$ implies $x \sim z$ (Transitivity).

In this case \sim is an equivalence relation of X . Given an equivalence relation of X , we write

$$(6.20) \quad [x] \equiv \{y \in X : x \sim y\} \in 2^X$$

for $x \in X$ and

$$(6.21) \quad X/\sim \equiv \{[x] : x \in X\} \subset 2^X.$$

Definition 6.27 (Quotient topology). Let \sim be an equivalence relation of a topological space X . Then the quotient topology of X with respect to \sim is the weakest topology such that

$$(6.22) \quad p : X \rightarrow X/\sim, x \mapsto [x]$$

is continuous. The mapping p is said to be the (canonical/natural) projection.

According to the definition, we see that $U \subset X/\sim$ is open if and only if $p^{-1}(U)$ is open.

As for this topology the following is elementary.

Theorem 6.28. *Let X and Y be topological spaces and \sim an equivalence relation of X . A mapping $f : X/\sim \rightarrow Y$ is continuous, if and only if $f \circ p : X \rightarrow Y$ is continuous.*

Equip $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ with the quotient topology.

Theorem 6.29. *The spaces $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ and $\mathcal{S}'_\infty(\mathbb{R}^n)$ are homeomorphic.*

Proof. According to Theorem 6.28, the mapping $\Phi : [f] \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n) \mapsto R(f) \in \mathcal{S}'_\infty(\mathbb{R}^n)$ is continuous. Let O be an open set in $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$. Then $O = p(\mathcal{O} + \mathcal{P}(\mathbb{R}^n))$ for some open set \mathcal{O} in $\mathcal{S}'(\mathbb{R}^n)$. Thus, $\Phi(O) = R(O)$ is open according to Theorem 6.19. \square

Finally to conclude this section, we prove another property of $\mathcal{S}_\infty(\mathbb{R}^n)$.

Theorem 6.30. *Assume that A is a bounded set in $\mathcal{S}_\infty(\mathbb{R}^n)$, that is,*

$$a_N = \sup_{f \in A} p_N(f) < \infty$$

for all $N \in \mathbb{N}$. Then A is a relatively compact set.

Proof. Since $\mathcal{S}_\infty(\mathbb{R}^n)$ is metrizable, we have only to show that any sequence $\{f_j\}_{j=1}^\infty$ in A has a convergent subsequence. Since $a_{N+1} < \infty$, we can use the Ascoli-Arzelà theorem to have a subsequence convergent in $\mathcal{V}_N(\mathbb{R}^n)$ from $\{f_j\}_{j=1}^\infty$. Cantor's diagonal argument yields a subsequence convergent in $\mathcal{S}(\mathbb{R}^n)$. Thus, A is relatively compact. \square

Remark 6.31. See [12, Theorem 2.2] for the extension of Theorem 6.30.

7. APPENDIX: COMPARISON WITH OTHER FUNCTION SPACES—HISTORY AND POSSIBLE EXTENSION OF OUR RESULTS

7.1. The characterization by means of the Peetre maximal operator. A recent trend on theory of function spaces such as Morrey spaces, Herz spaces and Orlicz spaces is to connect these spaces with the Littlewood-Paley decomposition. The idea of replacing $L^p(\mathbb{R}^n)$ spaces with other function spaces are expanded in [18, 30, 44, 72]. Such attempts are made for B_σ -function spaces, variable Lebesgue spaces and Orlicz spaces. See [24], [35, 42] and [43], respectively.

To recall the results, we use the notation based on [60]. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Define the (*nonhomogeneous*) *Besov-Morrey norm* by:

$$(7.1) \quad \|f\|_{\mathcal{N}_{pqr}^s} \equiv \|\psi(D)f\|_{\mathcal{M}_q^p} + \left(\sum_{j=1}^{\infty} 2^{jsr} \|\varphi_j(D)f\|_{\mathcal{M}_q^p}^r \right)^{1/r}$$

for $0 < q \leq p < \infty$, $0 < r \leq \infty$ and the (*nonhomogeneous*) *Triebel-Lizorkin norm* by:

$$(7.2) \quad \|f\|_{\mathcal{E}_{pqr}^s} \equiv \|\psi(D)f\|_{\mathcal{M}_q^p} + \left\| \left(\sum_{j=1}^{\infty} 2^{jsr} |\varphi_j(D)f|^r \right)^{1/r} \right\|_{\mathcal{M}_q^p}$$

for $0 < q \leq p < \infty$ and $0 < r \leq \infty$, where a natural modification is made in (7.1) and (7.2).

Now we can characterize our function spaces by means of the Peetre maximal operator in the spirit of [29, 30, 72].

Lemma 7.1. *Let ψ_j be as in Proposition 2.20. We define*

$$(7.3) \quad (\psi_j f)_*(x) \equiv \sup_{y \in \mathbb{R}^n} \frac{|f * \psi_j(y)|}{(1 + 2^j|x - y|)^N}$$

for $j \in \mathbb{N}_0$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. Let $0 < q < \infty$, $0 < r \leq \infty$ and $\varphi \in \mathcal{G}_q$.

(1) Let

$$(7.4) \quad N > \frac{n}{\min(1, q)} + n.$$

For any $f \in \mathcal{N}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$, we have

$$(7.5) \quad \left(\sum_{j=0}^{\infty} (\|2^{js}(\psi_j f)_*\|_{\mathcal{M}_q^\varphi})^r \right)^{\frac{1}{r}} \lesssim \|f\|_{\mathcal{N}_{\mathcal{M}_q^\varphi, r}^s}.$$

(2) Let

$$(7.6) \quad N > \frac{n}{\min(1, q, r)} + n.$$

For any $f \in \mathcal{E}_{\mathcal{M}_q^\varphi, r}^s(\mathbb{R}^n)$, we have

$$(7.7) \quad \left\| \left(\sum_{j=0}^{\infty} 2^{jst} ((\psi_j f)_*)^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^\varphi} \lesssim \|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^s}.$$

The proof is a direct consequence of Theorem 2.17.

Once such a characterization is obtained, we are in the position of applying a result in [30, Section 4] to obtain the decomposition results. However, the condition on L in (4.4) is milder than that obtained from [30, Theorem 4.5].

7.2. Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces. We can say that [27] is oldest among such attempts. The motivation was to obtain the solution starting from larger function spaces, which are called Besov Morrey spaces. Let $0 < q \leq p < \infty$, $0 < r \leq \infty$ and $s \in \mathbb{R}$. The Besov-Morrey space $\mathcal{N}_{pqr}^s(\mathbb{R}^n)$, which is a mixture of the Besov space and the Morrey space, emerged originally in the context of the analysis in the time-local solutions of the Navier-Stokes equations [27]. To investigate the time-local solutions of the equation H. Kozono and M. Yamazaki introduced the Besov-Morrey space $\mathcal{N}_{pqr}^s(\mathbb{R}^n)$ for the range $1 \leq q \leq p \leq \infty$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$; see [27, Definition 2.3]. Later on, this function space is extended and investigated intensively. A. Mazzucato investigated the decomposition of this function space [32]. She also investigated the pseudo-differential operators as well as the Besov-Morrey spaces on compact oriented Riemannian manifolds [33]. It is L. Tang and J. Xu that defined Triebel-Lizorkin-Morrey spaces $\mathcal{E}_{pqr}^s(\mathbb{R}^n)$ as well as they extended the parameters to the range $0 < q \leq p \leq \infty$, $0 < r \leq \infty$ and $s \in \mathbb{R}$ [68]. L. Tang and J. Xu investigated a different type of pseudo-differential operators as well [68, Section 4]; see [54] for further information. One also investigated various decompositions such as atomic decomposition, molecular decomposition and quarkonial decomposition with parameters $0 < q \leq p \leq \infty$, $0 < r \leq \infty$ and $s \in \mathbb{R}$; see [49] and [60, Theorems 4.1, 4.12, 5.3 and 5.9]. Xu and Fu characterized the Besov-Morrey space $\mathcal{N}_{pqr}^s(\mathbb{R}^n)$ and the Triebel-Lizorkin-Morrey space $\mathcal{E}_{pqr}^s(\mathbb{R}^n)$ when p and q are variable exponents

in [9]. Much was investigated for Morrey spaces a little earlier. Najafov considered Besov-Morrey spaces and Sobolev-Morrey spaces in [36] and [37], respectively. Najafov also considered the embedding results for Sobolev-Morrey spaces in [38]. Sawano and Tanaka considered the complex interpolation of Morrey spaces, Besov-Morrey spaces in [61] but there was a mistake in [61, Proposition 5.3]. The method introduced in the book [2] was not used in [61]. Yuan, Sickel and Yang overcame this problem in [97]. Other applications to PDE can be found in [23, 25, 26, 83].

7.3. Herz-type Besov spaces. Although Kozono and Yamazaki introduced Besov-Morrey spaces in 1994, much more was investigated for Herz spaces; Xu defined Herz-type Besov spaces. Let $1 < p, q < \infty$ and $\alpha \in \mathbb{R}$. We let $Q_0 \equiv [-1, 1]^n$ and $C_j \equiv [-2^j, 2^j]^n \setminus [-2^{j-1}, 2^{j-1}]^n$ for $j \in \mathbb{N}$. Recall that the Herz space $K_{p,q}^\alpha(\mathbb{R}^n)$ is the set of all measurable functions f for which the norm

$$\|f\|_{K_{p,q}^\alpha} \equiv \|\chi_{Q_0} \cdot f\|_{L^p} + \left(\sum_{j=1}^{\infty} (2^{j\alpha} \|\chi_{C_j} \cdot f\|_{L^p})^q \right)^{\frac{1}{q}}$$

is finite. Let $0 < r \leq \infty$ and $s \in \mathbb{R}$. The Herz-type Besov space $K_{p,q}^\alpha B_r^s(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{K_{p,q}^\alpha B_r^s} \equiv \begin{cases} \|\theta(D)f\|_{K_{p,q}^\alpha} + \left(\sum_{j=1}^{\infty} 2^{jsr} \|\tau_j(D)f\|_{K_{p,q}^\alpha}^r \right)^{\frac{1}{r}} & (r < \infty), \\ \|\theta(D)f\|_{K_{p,q}^\alpha} + \sup_{j \in \mathbb{N}} 2^{js} \|\tau_j(D)f\|_{K_{p,q}^\alpha} & (r = \infty) \end{cases}$$

is finite. Xu considered the boundedness property of the Fourier multiplier in [73] for Herz-type Triebel-Lizorkin spaces and proved the boundedness property of the lift operator as well as the embedding property of the Schwartz class in [74]. The boundedness property of the pointwise multiplier, described in Section 5, is obtained in [75, 79, 84]. Xu proved the boundedness property of the pseudo-differential operators in [76, 79, 84]. We say that a quasi-normed space X is called to be admissible, if for every compact subset $E \subset X$ and for every $\varepsilon > 0$, there exists a continuous map $T : E \rightarrow X$ such that $T(E)$ is contained in a finite dimensional subset of X and $\|Tx - x\|_X \leq \varepsilon$ for all $x \in E$. Xu characterized the Herz-type Besov spaces by means of the Peetre maximal operator in [77, 78] and used this characterization to prove the admissibility in [80]. Xu obtained the atomic decomposition, the molecular decomposition, and the wavelet decomposition in [82]. We can find applications of Herz-type Triebel-Lizorkin spaces to partial differential equations, more precisely, to the Beal-Kato-Majda type and the Moser type inequalities in [81]. Dong and Xu considered the case when α and p are variable exponents in [4]. Shi and Xu considered Herz-type Triebel-Lizorkin spaces with α and p variable in [65]. Likewise we can consider Herz-Morrey spaces. Recall that the Herz-Morrey space $K_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ is the set of all measurable functions f for which the norm

$$\|f\|_{MK_{p,q}^{\alpha,\lambda}} \equiv \sup_{L \in \mathbb{N}_0} 2^{-L\lambda} \left(\|\chi_{Q_0} \cdot f\|_{L^p} + \left(\sum_{j=1}^L (2^{j\alpha} \|\chi_{C_j} \cdot f\|_{L^p})^q \right)^{\frac{1}{q}} \right)$$

is finite. Dong and Xu considered the case when α and p are variable exponents in [5].

7.4. Besov-type spaces and Triebel-Lizorkin type spaces. Yang and Yuan investigated Besov type spaces and Triebel-Lizorkin type spaces in [85, 86]. Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $\rho \geq 0$.

For $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ one defines the homogeneous Besov-type norm and the homogeneous Triebel-Lizorkin type norm by:

$$\|f\|_{\dot{B}_{pq}^{s,\rho}} \equiv \sup_{Q \in \mathcal{D}} \frac{1}{|Q|^\rho} \left(\sum_{j=-\log_2 \ell(Q)}^{\infty} 2^{sqj} \left(\int_Q |\tau_j(D)f(x)|^p dx \right)^{\frac{q}{p}} \right)^{\frac{1}{q}},$$

$$\|f\|_{\dot{F}_{pq}^{s,\rho}} \equiv \sup_{Q \in \mathcal{D}} \frac{1}{|Q|^\rho} \left(\int_Q \left(\sum_{j=-\log_2 \ell(Q)}^{\infty} 2^{sqj} |\tau_j(D)f(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}},$$

respectively. The spaces $B_{pq}^{s,\rho}(\mathbb{R}^n)$, $F_{pq}^{s,\rho}(\mathbb{R}^n)$ stand for linear spaces of functions $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ for which the quantities $\|f\|_{B_{pq}^{s,\rho}}$, $\|f\|_{F_{pq}^{s,\rho}} < \infty$ respectively. The notation $\dot{A}_{pq}^{s,\rho}$ stands for either $\dot{B}_{pq}^{s,\rho}$ or $\dot{F}_{pq}^{s,\rho}$. The inhomogeneous spaces are defined analogously. To connect these scales with ours, we prove the following proposition:

Proposition 7.2. *Let $0 < q < \infty$, $0 < r \leq \infty$ and $\varphi \in \mathcal{G}_q$ satisfies (2.4). For $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$. define*

$$\|f\|_{\dot{F}_{qr}^{s,\varphi}} \equiv \sup_{Q \in \mathcal{D}} \varphi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q \left(\sum_{j=-\log_2 \ell(Q)}^{\infty} 2^{sqj} |\tau_j(D)f(x)|^r \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}}.$$

Then $\|f\|_{\mathcal{E}_{\mathcal{M}_s^{\varphi,r}}^s} \sim \|f\|_{\dot{F}_{qr}^{s,\varphi}}$ for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$.

Sawano, Yang and Yuan proved that the above scale \mathcal{E} turned out to be Triebel-Lizorkin type spaces [62]. Proposition 7.2 is an analogy to this fact. In view of [62, Theorem 1.1], the counterpart for generalized Besov-Morrey spaces is not available.

Proof of Proposition 7.2. To this end, comparing these norms, we have only to show

$$(7.8) \quad \varphi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q \left(\sum_{j=-\infty}^{-\log_2 \ell(Q)} 2^{sqj} |\tau_j(D)f(x)|^r \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}} \lesssim \|f\|_{\dot{F}_{qr}^{s,\varphi}}.$$

Similarly to Example 2.18, we have

$$(7.9) \quad \|\tau_j(D)f\|_{L^\infty} \leq \varphi(2^{-j})^{-1} \|f\|_{\dot{F}_{qr}^{s,\varphi}}.$$

Thus, by combining Proposition 2.7 and (7.9), we obtain (7.8). \square

The space $\mathcal{E}_{pq2}^0(\mathbb{R}^n)$ with $0 < q \leq p < \infty$ has many other equivalent norms. In [54], it was shown that this function space $\mathcal{E}_{pq2}^0(\mathbb{R}^n)$ is equivalent to the Hardy-Morrey space defined by Jia and Wang in [22]. See [1, 20, 22, 63] for more about Hardy Morrey spaces. In [32, Theorem 4.2], Mazzucato proved that the Triebel-Lizorkin space $\mathcal{E}_{pq2}^0(\mathbb{R}^n)$ with $1 < q \leq p < \infty$ is equivalent to the Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$. This type of norm equivalence is extended to many other function spaces in [30] and many authors applied this equivalence to PDEs. See [19] for the boundedness of singular integral operators. By combining the wavelet characterization of Besov-Morrey spaces [48, Theorem 1.5] and the embedding criterion of the corresponding sequence space [16, Theorem 3.2], Haroske and Skrzypczak obtained the necessary and sufficient conditions on the parameters $p_0, q_0, r_0, s_0, p_1, q_1, r_1$ and s_1 for the embedding $\mathcal{N}_{p_0 q_0 r_0}^{s_0}(\mathbb{R}^n) \hookrightarrow \mathcal{N}_{p_1 q_1 r_1}^{s_1}(\mathbb{R}^n)$ to hold. In the context of the generalized Triebel-Lizorkin Morrey spaces, we can improve [63, Theorem 5.1] as follows:

Proposition 7.3. *Let $1 \leq q \leq p < \infty$ and $0 < r < q$. Then by definining*

$$(7.10) \quad \varphi(t) \equiv \left[\log \left(2 + \frac{1}{t} \right) \right]^{-1/\min(1,r)} \quad (t > 0),$$

we have $\|f\|_{\mathcal{E}_{\mathcal{M}_q^\varphi, r}^0} \lesssim \|f\|_{\mathcal{E}_{pq\infty}^{n/p}}$ for all $f \in \mathcal{E}_{pq\infty}^{n/p}(\mathbb{R}^n)$.

Proof. According to Theorems 4.4 and 4.5, we have only to prove

$$\mathbf{e}_{pq\infty}^{n/p}(\mathbb{R}^n) \equiv \mathbf{e}_{\mathcal{M}_q^\psi, \infty}^{n/p}(\mathbb{R}^n) \subset \mathbf{e}_{\mathcal{M}_q^\varphi, r}^0(\mathbb{R}^n),$$

where $\psi(t) \equiv t^{n/p}, t > 0$. This amounts to showing that

$$(7.11) \quad \begin{aligned} & \varphi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q \left(\sum_{\nu=0}^{\infty} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^r \right)^{q/r} dx \right)^{1/q} \\ & \lesssim |Q|^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q \left(\sup_{\nu \in \mathbb{N}_0} 2^{\nu n/p} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^q \right) dx \right)^{1/q} \end{aligned}$$

for all complex sequences $\{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$.

If $\ell(Q) \geq 1$, then (7.11) is easy to prove; we just combine

$$\varphi(\ell(Q)) \lesssim |Q|^{1/p}, \quad \sum_{\nu=0}^{\infty} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^r \lesssim \sup_{\nu \in \mathbb{N}_0} 2^{\nu n r/p} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^r.$$

If $\ell(Q) \leq 1$, we shall prove;

$$(7.12) \quad \begin{aligned} & \varphi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q \left(\sum_{\nu=j_Q}^{\infty} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^r \right)^{q/r} dx \right)^{1/q} \\ & \lesssim |Q|^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q \left(\sup_{\nu \in \mathbb{N}_0} 2^{\nu n/p} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^q \right) dx \right)^{1/q} \end{aligned}$$

and

$$(7.13) \quad \begin{aligned} & \varphi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q \left(\sum_{\nu=0}^{j_Q} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^r \right)^{q/r} dx \right)^{1/q} \\ & \lesssim |Q|^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q \left(\sup_{\nu \in \mathbb{N}_0} 2^{\nu n/p} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^q \right) dx \right)^{1/q}. \end{aligned}$$

We have (7.12) since, for any $Q \in \mathcal{D}$,

$$\varphi(\ell(Q)) \lesssim 1, \quad \sum_{\nu=j_Q}^{\infty} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^r \lesssim |Q|^{1/p} \sup_{\nu \in \mathbb{N}_0} 2^{\nu n r/p} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^r.$$

As for (7.13), we write $m(\nu) \in \mathbb{Z}^n, \nu \leq j_Q$ for the unique element $m \in \mathbb{Z}^n$ such that $Q_{\nu m} \supset Q$. we use the Minkowski inequality and the triangle inequality to have;

$$\begin{aligned}
& \varphi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q \left(\sum_{\nu=0}^{j_Q} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^r \right)^{q/r} dx \right)^{1/q} \\
& \leq \varphi(\ell(Q)) \left(\sum_{\nu=0}^{j_Q} \left(\frac{1}{|Q|} \int_Q \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^q dx \right)^{r/q} \right)^{1/r} \\
& = \varphi(\ell(Q)) \left(\sum_{\nu=0}^{j_Q} \left(\frac{1}{|Q_{m(\nu)\nu}|} \int_{Q_{m(\nu)\nu}} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^q dx \right)^{r/q} \right)^{1/r} \\
& \leq \varphi(\ell(Q)) \left(\sum_{\nu=0}^{j_Q} |Q_{m(\nu)\nu}|^{\frac{r}{p} - \frac{r}{q}} \left(\int_{Q_{m(\nu)\nu}} \left(\sup_{\nu \in \mathbb{N}_0} 2^{\nu n/p} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}}(x) \right|^q \right)^{r/q} dx \right)^{1/r} \right) \\
& \lesssim \varphi(\ell(Q)) (j_Q)^{1/r} \|f\|_{\mathcal{E}_{pq\infty}^{n/p}} \lesssim \|f\|_{\mathcal{E}_{pq\infty}^{n/p}}.
\end{aligned}$$

Thus, (7.11) is proved. \square

A couple of remarks on Proposition 7.3 may be in order.

Remark 7.4. Let $1 < q \leq p < \infty$.

- (1) Let $r = 1$ in Proposition 7.3. The authors in [63] showed that

$$\|f\|_{\mathcal{M}_1^\varphi} \lesssim \|(1 - \Delta)^{n/2p} f\|_{\mathcal{M}_q^p}$$

for all $f \in \mathcal{M}_q^p(\mathbb{R}^n)$ with $\Delta f \in \mathcal{M}_q^p(\mathbb{R}^n)$. Since

$$\|f\|_{\mathcal{M}_1^\varphi} \lesssim \|f\|_{\mathcal{E}_{\mathcal{M}_q^1,1}^s}, \quad \|f\|_{\mathcal{E}_{pq\infty}^{n/p}} \lesssim \|(1 - \Delta)^{n/2p} f\|_{\mathcal{M}_q^p}$$

for all $f \in \mathcal{M}_q^p(\mathbb{R}^n)$ with $\Delta f \in \mathcal{M}_q^p(\mathbb{R}^n)$, Proposition 7.3 improves [63, Theorem 5.1].

- (2) According to the necessary and sufficient condition in [7, Theorem 2], one can not have

$$\|f\|_\infty \lesssim \|(1 - \Delta)^{n/2p} f\|_{\mathcal{M}_q^p}.$$

- (3) One can not replace $\min(1, r)$ by 1 in (7.10) when $r \in (0, 1)$. Assume to the contrary that this is possible. Let $x_0 = (3/2, 0, \dots, 0)$ and $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that $\chi_{Q(x_0, 1/100)} \leq \eta \leq \chi_{Q(x_0, 1/10)}$. Choose $\tau \in C_c^\infty(\mathbb{R}^n)$ so that $\chi_{Q(2) \setminus Q(1)} \leq \tau \leq \chi_{Q(2+1/10) \setminus Q(9/10)}$. Then for

$$f_N \equiv \sum_{l=1}^N \mathcal{F}^{-1} \eta_k \quad (N = 1, 2, \dots),$$

one has

$$\|f_N\|_{\mathcal{E}_{\mathcal{M}_q^1, r}^0} \gtrsim \varphi(2^{-N}) \left(\frac{1}{|Q|} \int_Q \left(\sum_{k=1}^N |\mathcal{F}^{-1} \eta_k(x)|^r \right)^{q/r} dx \right)^{1/q} \gtrsim \varphi(2^{-N}) N^{1/r}.$$

Meanwhile, $\|f_N\|_{\mathcal{E}_{pq\infty}^{n/p}} \lesssim 1$. Thus, this is a contradiction since N is arbitrary.

Haroske and Skrzypczak work also with the setting of bounded open sets Ω and prove similar results; the necessary and sufficient conditions on the parameters $p_0, q_0, r_0, s_0, p_1, q_1, r_1$ and s_1 for the embedding $\mathcal{N}_{p_0 q_0 r_0}^{s_0}(\Omega) \hookrightarrow \mathcal{N}_{p_1 q_1 r_1}^{s_1}(\Omega)$ to hold. Here for the definition of the function

space $\mathcal{N}_{pqr}^s(\Omega)$ can be found in [51, Definition 5.1] and [17, Definition 2.7]. See [52, 57, 93] for more results on the Sobolev embedding theorem.

In [71], Triebel introduced the so called local space $\mathcal{L}^r A_{pq}^s(\mathbb{R}^n)$. See [71] and [92, Section 3] for the definition. In [92], Yang, Sickel and Yuan proved that the scale $A_{pq}^{s,\tau}(\mathbb{R}^n)$ comes about naturally as a result of the localization of $A_{pq}^s(\mathbb{R}^n)$. The local means are considered in [45]. Yang, Yuan and Zhuo investigated the boundedness property of the Fourier multiplier precisely in [94]. Yang and Yuan characterized $A_{pq}^{s,\rho}(\mathbb{R}^n)$ in terms of the Peetre maximal operator in [87]. See [89, 96] for more recent advances in Triebel-Lizorkin type spaces, and see [17, 51] for the extension of this scale to domains. We refer to [66, 67, 91, 90] for an exhaustive account of these function spaces as well as of results on decompositions.

7.5. Orlicz–Morrey spaces and Musielak–Orlicz Triebel–Lizorkin-type spaces. Recall that for a function $\varphi : \mathbb{R}^n \times [0, \infty) \rightarrow [0, \infty)$, and a measurable function defined on \mathbb{R}^n , the Musielak–Orlicz norm is given by:

$$\|f\|_{L^\varphi} \equiv \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \varphi \left(x, \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

Here and below we let $I_1, I_2, i_1, i_2 > 0$, $q_1, q_2, \delta_1, \delta_2 > 1$ and assume that $\varphi_1, \varphi_2 : \mathbb{R}^n \times [0, \infty) \rightarrow [0, \infty)$ are functions satisfying the upper and lower type conditions

$$s^{i_k} \varphi_k(x, t) \lesssim \varphi_k(x, st) \lesssim s^{I_k} \varphi_k(x, t) \quad (x \in \mathbb{R}^n, s \geq 1, t > 0, k = 1, 2),$$

the uniformly A_∞ -condition

$$\frac{1}{|Q|} \int_Q \varphi_k(x, t) dx \left(\frac{1}{|Q|} \int_Q \varphi_k(x, t)^{1-q_k} dx \right)^{q_k-1} \lesssim 1 \quad (t \geq 0, Q \in \mathcal{Q})$$

and the reverse Hölder condition

$$\left(\frac{1}{|Q|} \int_Q \varphi_k(x, t)^{\delta_k} dx \right)^{1/\delta_k} \leq C \frac{1}{|Q|} \int_Q \varphi_k(x, t) dx;$$

see [95, p. 96]. In view of Proposition 7.2, we can say that $\mathcal{E}_{\mathcal{M}_s^{\varphi}, r}^s(\mathbb{R}^n)$ is the Musielak–Orlicz Triebel–Lizorkin-type space of $\dot{F}_{\varphi_1, \varphi_2, q}^{s, \tau}(\mathbb{R}^n)$ defined in [95, Definition 2.1]. Let us recall the definition.

Definition 7.5. Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $q \in (0, \infty]$. Then define the Musielak–Orlicz Triebel–Lizorkin-type space $\dot{F}_{\varphi_1, \varphi_2, q}^{s, \tau}(\mathbb{R}^n)$ as the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{\dot{F}_{\varphi_1, \varphi_2, q}^{s, \tau}(\mathbb{R}^n)} \equiv \sup_{Q \in \mathcal{D}} \frac{1}{(\|\chi_Q\|_{L^{\varphi_1}})^\tau} \left\| \left[\chi_Q \sum_{j=j_Q}^{\infty} (2^{js} |\tau_j(D)f|)^q \right]^{\frac{1}{q}} \right\|_{L^{\varphi_2}}$$

is finite.

In this sense, the results for $\mathcal{E}_{\mathcal{M}_s^{\varphi}, r}^s(\mathbb{R}^n)$ can be covered by [95]. For example, Proposition 3.2 can be understood as the inhomogeneous version of [95, Proposition 2.19]. Observe that [95, Theorem 3.1] characterizes Musielak–Orlicz Triebel–Lizorkin-type spaces by means of the Peetre maximal operator given below. Our atomic decomposition results, Theorems 4.4 and 4.5 correspond to [95, Theorem 5.1]. By using the idea of [95, Theorem 6.9] or [50], we can prove the pseudo-differential operators with symbol in S^0 is bounded in $\mathcal{A}_{\mathcal{M}_q^{\varphi}, r}^s(\mathbb{R}^n)$.

Let us check that Musielak–Orlicz Triebel–Lizorkin-type spaces come from (one of) generalized Orlicz Morrey spaces. To the best knowledge of the authors, there exists three generalized Orlicz–Morrey spaces.

Definition 7.6. Let $\Phi \in \mathbb{R}^n \times [0, \infty) \rightarrow [0, \infty)$ and $\varphi : \mathcal{Q} \rightarrow [0, \infty)$ be suitable functions.

- (1) For a cube $Q \in \mathcal{Q}$ define the (φ, Φ) -average over Q of the measurable function f by

$$\|f\|_{(\varphi, \Phi); Q} \equiv \inf \left\{ \lambda > 0 : \frac{\varphi(Q)}{|Q|} \int_Q \Phi \left(x, \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

Define the generalized Orlicz-Morrey space $\mathcal{L}^{\varphi, \Phi}(\mathbb{R}^n)$ to be a Banach space equipped with the norm

$$\|f\|_{\mathcal{L}^{\varphi, \Phi}} \equiv \sup_{Q \in \mathcal{Q}} \|f\|_{(\varphi, \Phi); Q}.$$

- (2) For a cube $Q \in \mathcal{Q}$ define the Φ -average over Q of the measurable function f by

$$\|f\|_{\Phi; Q} \equiv \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left(x, \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

Define the generalized Orlicz-Morrey space $\mathcal{M}_{\varphi, \Phi}(\mathbb{R}^n)$ to be a Banach space equipped with the norm

$$\|f\|_{\mathcal{M}_{\varphi, \Phi}} \equiv \sup_{Q \in \mathcal{Q}} \varphi(Q) \|f\|_{\Phi; Q}.$$

- (3) Define the generalized Orlicz-Morrey space $\mathcal{Z}_{\varphi, \Phi}(\mathbb{R}^n)$ to be a Banach space equipped with the norm

$$\|f\|_{\mathcal{Z}_{\varphi, \Phi}} \equiv \sup_{Q \in \mathcal{Q}} \varphi(Q) \|\chi_Q f\|_{L^\Phi}.$$

The spaces $\mathcal{L}^{\varphi, \Phi}(\mathbb{R}^n)$, $\mathcal{M}_{\varphi, \Phi}(\mathbb{R}^n)$ and $\mathcal{Z}_{\varphi, \Phi}(\mathbb{R}^n)$ are defined by Nakai in [41] (with Φ independent of x), by Sawano, Sugano and Tanaka in [58] (with Φ independent of x) and by Deringoz, Guliyev and Samko in [3, Definition 2.3], respectively. According to the examples in [11], we can say that the scales \mathcal{L} and \mathcal{M} are different and that \mathcal{M} and \mathcal{Z} are different. However, it is not known that \mathcal{L} and \mathcal{Z} are different.

In Proposition 3.2, we rephrased the notion of generalized Triebel-Lizorkin-Morrey spaces in the language of Triebel-Lizorkin type spaces. We can do the vice versa as the following lemma implies.

Lemma 7.7. [95, Theorem 4.1] *Assume that τ satisfy*

$$0 \leq \tau < \frac{i_1(\delta_2 - 1)}{q_1 I_2 \delta_2}.$$

Then

$$\|f\|_{\dot{F}_{\varphi_1, \varphi_2, q}^{s, \tau}(\mathbb{R}^n)} \sim \sup_{Q \in \mathcal{D}} \frac{1}{(\|\chi_Q\|_{L^{\varphi_1}})^\tau} \left\| \left[\chi_Q \sum_{j=-\infty}^{\infty} (2^{js} |\tau_j(D)f|)^q \right]^{\frac{1}{q}} \right\|_{L^{\varphi_2}}.$$

Therefore, by setting

$$\varphi(x, r) \equiv \frac{1}{(\|\chi_{Q(x, r)}\|_{L^{\varphi_1}})^\tau}, \quad \Phi(x, r) \equiv \varphi_2(x, r),$$

we can say that Musielak-Orlicz Triebel-Lizorkin-type space $\dot{F}_{\varphi_1, \varphi_2, q}^{s, \tau}(\mathbb{R}^n)$ come from the generalized Orlicz Morrey space $\mathcal{Z}_{\varphi, \Phi}(\mathbb{R}^n)$.

7.6. Hausdorff Besov-type spaces and Hausdorff Triebel-Lizorkin type spaces. Let us recall the definition of Besov-Hausdorff spaces and Triebel-Lizorkin-Hausdorff spaces, which are the predual spaces of Besov-type spaces and Triebel-Lizorkin type spaces. Let $E \subset \mathbb{R}^n$ and $d \in (0, n]$. The d -dimensional Hausdorff capacity of E is defined by

$$H^d(E) \equiv \inf \left\{ \sum_j r_j^d : E \subset \bigcup_j B(x_j, r_j) \right\},$$

where the infimum is taken over all covers $\{B(x_j, r_j)\}_{j=1}^\infty$ of E by countable families of open balls. It is well known that H^d is monotone, countably subadditive and vanishes on empty sets. Moreover, the notion of H^d can be extended to $d = 0$. In this case, H^0 has the property that for all nonempty sets $E \subset \mathbb{R}^n$, $H^0(E) \geq 1$, and $H^0(E) = 1$ if and only if E is bounded.

Definition 7.8 (Choquet integral). For any function $f : \mathbb{R}^n \mapsto [0, \infty]$, the *Choquet integral* of f with respect to H^d is defined by

$$\int_{\mathbb{R}^n} f dH^d \equiv \int_0^\infty H^d(\{x \in \mathbb{R}^n : f(x) > \lambda\}) d\lambda.$$

This functional is not sublinear, so sometimes we need to use an equivalent integral with respect to the d -dimensional dyadic Hausdorff capacity \tilde{H}^d , which is sublinear.

To define the spaces, we also need the nontangential maximal operator.

Definition 7.9 (Nontangential maximal operator). Let $\mathbb{R}_+^{n+1} \equiv \mathbb{R}^n \times (0, \infty)$. For any measurable function ω on \mathbb{R}_+^{n+1} and $x \in \mathbb{R}^n$, we define its *nontangential maximal function* $N\omega(x)$ by setting $N\omega(x) \equiv \sup_{|y-x|<t} |\omega(y, t)|$.

Definition 7.10. Let $p \in (1, \infty)$ and $s \in \mathbb{R}$.

- (1) If $q \in [1, \infty)$ and $\tau \in [0, \frac{1}{\max(p, q)'}]$, the *Besov-Hausdorff space* $B\dot{H}_{p, q}^{s, \tau}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{B\dot{H}_{p, q}^{s, \tau}(\mathbb{R}^n)} \equiv \inf_\omega \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \|\tau_j(D)f \cdot [\omega(\cdot, 2^{-j})]^{-1}\|_{L^p(\mathbb{R}^n)}^q \right\}^{\frac{1}{q}}$$

is finite, where ω runs over all nonnegative Borel measurable functions on \mathbb{R}_+^{n+1} such that

$$(7.14) \quad \int_{\mathbb{R}^n} [N\omega(x)]^{\max(p, q)'} dH^{n\tau \max(p, q)'}(x) \leq 1$$

and with the restriction that for any $j \in \mathbb{Z}$, $\omega(\cdot, 2^{-j})$ is allowed to vanish only where $\tau_j(D)f$ vanishes.

- (2) If $q \in (1, \infty)$ and $\tau \in [0, \frac{1}{\max(p, q)'}]$, the *Triebel-Lizorkin-Hausdorff space* $F\dot{H}_{p, q}^{s, \tau}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{F\dot{H}_{p, q}^{s, \tau}(\mathbb{R}^n)} \equiv \inf_\omega \left\| \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} |\tau_j(D)f \cdot [\omega(\cdot, 2^{-j})]^{-1}|^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}$$

is finite, where ω runs over all nonnegative Borel measurable functions on \mathbb{R}_+^{n+1} such that ω satisfies (7.14) and with the restriction that for any $j \in \mathbb{Z}$, $\omega(\cdot, 2^{-j})$ is allowed to vanish only where $\tau_j(D)f$ vanishes.

Yang and Yuan proved that these spaces are realized as the dual space of a subspace of $A_{pq}^{s,\rho}(\mathbb{R}^n)$; see [88]. It is proved in [98] that $FH_{p,q}^{s,\rho}(\mathbb{R}^n)$ covers the predual space of Morrey spaces.

7.7. Function spaces in the metric measure settings. These function spaces carry over to the weighted settings or to the metric measure setting. For $x \in \mathbb{R}^n$ and $r > 0$ or more generally for x in the metric measure space (X, d) and $r > 0$, we write

$$B(x, r) \equiv \{y \in X : d(x, y) < r\}.$$

Based on the Morrey space defined in [59], Sawano and Tanaka considered Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces in \mathbb{R}^n equipped with a Radon measure satisfying

$$(7.15) \quad \mu(B(x, r)) \leq Cr^D$$

for some $0 < D \leq n$; see [61]. Probably, the theory can be generalized to the setting of geometrically doubling measure space equipped with a Radon measure satisfying (7.15). See [10, 31] for some results in such setting. In particular, in [10], the authors developed the theory of $H^p(\mathbb{R}^n)$ spaces with $p \in (0, 1)$ with nondoubling measures, which was supposed to be difficult. Izuki, Sawano and Tanaka investigated the weighted spaces with weights in A_{loc}^∞ [21]. These types of function spaces will shed light on because $\mathcal{S}(\mathbb{R}^n)$ are not dense in them.

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